Computation of A_∞ algebras in group cohomology

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Abstract

The applicability of A_∞-structures, as introduced by Stasheff (1963) in the setting of algebra, or more specifically in representation theory has gradually grown. With Keller (2002, 2001) and Lu, Palmieri, Wu, and Zhang (2007, 2006), the benefits of A_∞-algebra structures to the theory of representations and of cohomology of various classes of algebras have been made explicit. By the results in these papers, there is the potential to use A_∞-structures in modular group cohomology – the group cohomology A_∞-algebra would determine the corresponding group ring completely.

However, the calculational techniques available do not lend themselves easily to calculation of these structures, nor are there particularly many examples at hand. The only examples that do exist in deeper treatment for modular finite group cohomology are the finite cyclic groups.

Using the methods of Saneblidze and Umble, the A_∞-structures on the cohomology rings of finite cyclic groups would induce structures on all finite abelian groups. However, even though the Saneblidze-Umble diagonal gives a completely determined structure, it is not necessarily easily comprehended in specific cases.

In this thesis, steps are taken to remedy these problems – we give computer implementations in Haskell and in Magma of the calculation of the Saneblidze-Umble diagonal terms and of blackbox calculation of A_∞ operations with the aim of using both in the calculation of A_∞-structures on group cohomology rings. This is demonstrated with a systematic re-computation of the results from Madsen (2002) on an A_∞-structure on $H^*(C_n, \mathbb{F}_p)$, including technical conditions for reduction of the computational load. We further give a few specific results on the A_∞-structure of $H^*(C_n \times C_m; \mathbb{F}_p)$.

Zusammenfassung

Die Anwendungsmöglichkeiten der A_∞-Strukturen, die von Stasheff (1963) eingeführt wurden, haben sowohl in der Algebra als auch in der Darstellungstheorie stetig zugenommen. Mit Keller (2002, 2001) und Lu, Palmieri, Wu, and Zhang (2007, 2006) wurden die Vorteile der A_∞-Strukturen für das Studium der Darstellungstheorie und der Kohomologieringe verschiedener Klassen von Algebren deutlich. Durch diese Ergebnisse eröffnet sich die Möglichkeit, A_∞-Strukturen auch in der modularen Gruppenkohomologie einzusetzen – die A_∞-Struktur eines Kohomologieringes bestimmt die ursprüngliche Gruppenalgebra vollständig.

Allerdings sind die am weitesten verbreiteten Berechnungstechniken für A_∞-Strukturen nicht besonders leicht zu nutzen. Weiterhin gibt es fast keine Beispiele von A_∞-Strukturen die in der Gruppenkohomologie vorkommen könnten. Die einzigen vollständig beschriebenen Beispiele, die überhaupt existieren, sind jene für die zyklischen Gruppen.

Mit den Methoden von Saneblidze und Umble werden von den schon bekannten A_∞-Strukturen auf den Kohomologieringen der endlichen zyklischen Gruppen induzierte A_∞-Strukturen auf den Kohomologieringen der endlichen abelschen Gruppen erzeugt. Obgleich die Saneblidze-Umble-Diagonale eine vollständige Struktur bestimmt, ist sie in spezifischen Fällen nicht unbedingt leicht zu beschreiben.

In dieser Dissertation werden mehrere Schritte unternommen um diese Probleme anzugehen. Wir geben Computerimplementierungen in Haskell und Magma an für die Berechnung der Terme der Saneblidze-Umble-Diagonale und für die Black Box-Berechnungen der A_∞-Strukturen der Gruppenkohomologieringe. Diese werden dann genutzt, um die Berechnungen von Madsen (2002) auf eine systematischere Weise zu wiederholen. Weiterhin geben wir einige spezifische Ergebnisse zur A_∞-Struktur von $H^*(C_n \times C_m; \mathbb{F}_p)$.

To all my teachers. Past, present and future.

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I owe my family, and in particular my wife Susanne Vejdemo, deep thanks for their support and interest in my undertakings – even when following my passion leads me to move abroad.

Contents

Chapter 1

Introduction

> Epigrams scorn detail and make a point: They are a superb high-level documentation.
>
> Alan J. Perlis

1.1 Group cohomology and homological algebra

Consider a topological space. We have a few tools available to extract information about it – and among the first we learn about we find homology. By approximating the space with another space with a particularly good structure, we can break it down into well-understood pieces and by plain linear algebra extract surprisingly much about the space as such.

Once this got started, the scope for what entities homology can tackle was gradually expanded. At first, during the first part of the 20th century, the main mode of study was building topological spaces out of the entities concerned and studying these. Hence, we could for instance take a group G and build a space as follows. First, we put in one single 0-cell. Then, we add a loop beginning and ending in this cell for each group element in G. Then, we paste a disc in each relation among the elements – if $g_1 g_2 \dots g_n = e$, then one disc is pasted with boundary the concatenation of the paths $g_1 \dots g_n$. Adding further higher dimensional cells to kill all relations that arise, we end up with the covering space BG of the group. This is the first Eilenberg-Mac Lane-space $K(G; 1)$, and it has the very nice property that its fundamental group is isomorphic to G, by design, and all higher homotopy groups vanish.

However, the homology and cohomology of BG exhibit rich structure.

Gradually, the method of study shifted from algebraic topology to homological algebra. Instead of taking a space and generating a chain complex that subsequently is studied, we just generate the chain complex. This can be done in ways that emulate the topological origins closely enough for the study to be basically the same – while at the same time yielding a process completely algebraic in nature, albeit inspired by topology.

Thus, we introduce the idea of a resolution of a module M. A resolution is a differential graded module pM with a nice structure to each homogenous component – commonly flat, free or projective – such that the cohomology $H^*(pM)$ of the resolution is isomorphic to M as differential graded modules. We can then use the niceness of the components of pM to facilitate study of M itself.

Thus, a resolution is to a module what a chain complex is to a topological space – we capture some of the salient properties in an object for which we have many further tools. In particular, we can develop the study of chain complexes into a field that allows us to adapt the topological methods to non-topological questions.

The Ext-functor arises in this context as the algebraic reformulation of the study of the cohomology of the Eilenberg-Mac Lane-space. We write $\mathrm{Ext}^*_R(M,N)$ for the homology of the chain complex $\mathrm{Hom}((pM)_*,N)$. It turns out that $H^*(K(G;1);k) = \mathrm{Ext}^*_{kG}(k,k)$ where k is some field of coefficients, and also viewed as the trivial kG-module. We denote this by $H^*(G,k)$ for simplicity.

Low-dimensional components of $H^*(G,k)$ encode many interesting properties both of G and of kG. For one tangible example, the number of essentially different extensions M

$$0 \to L \to M \to N \to 0$$

of kG-modules is given by the dimension of $\mathrm{Ext}^1_{kG}(N,L)$.

The Ext-functor as such is studied in many different areas – what makes the particular case of group cohomology interesting is that we gain alternative routes of attack on the posed problems. We can use homological algebra methods to take on group theoretic problems, and we can use group theory to tackle homological algebra problems in this specific setting. It gives us a testing ground for the use of homological algebra in representation theory – while we would want to use the methods developed in any module category, we have a strong control on the structures encountered in the category of kG-modules for the right choices of k and G.

1.2 Associativity up to homotopy

From a pointed topological space (X, x_0) we can form the loop space ΩX, by considering all loops – maps from the pointed circle $(S^1, *) \to (X, x_0)$. Two loops can be composed by running through first one and then the other. As such, the loop space ΩX with this composition operation fails associativity, but by considering equivalence classes of loops under homotopy, we retrieve a group, the fundamental group $\pi_1(X, x_0)$.

The central point in this construction is where we take the quotient with the homotopy relation. By doing this, we get associativity of the operation. We do, however, throw away a lot of the information inherent in the loop space.

One way of retaining this information is to choose homotopies for all associativity conditions we resolve. We know that these exist, since composition in ΩX is associative up to homotopy. However, once we have chosen homotopies for the associativity condition, these homotopies in turn interact, and we find higher conditions involving the chosen homotopies and the composition operation prompting a choice of new homotopies to deal with the derived conditions.

The original context for A_∞-structures was in the doctoral thesis of Jim Stasheff (1963). There, previously existing criteria to decide whether a particular topological space X can be realized as the loop space ΩY of some other topological space Y were refined. In Stasheff's formulation, the existence of an A_∞-structure on a topological space with a continuous composition map implies that it is in fact a loop space.

The structures there defined find additional uses outside the purely topological applications. One important method in homological algebra and algebraic topology is to take a large chain complex described in some easily handled manner, and cut it down in size to some smaller chain complex that makes computations more feasible. One example comes from group cohomology – the bar, or standard, resolution is easy to describe, but the ranks of the components P_n grow exponentially with n. For most groups, the minimal resolution will be significantly smaller than the standard resolution, and computationally much will be won by cutting the resolution down in size. Various methods exist for doing this in specific cases, and most of them consist of methods to find strong deformation retract data from the larger resolution to the smaller one.

When the resolution comes with an algebraic structure beyond being a dg-module – when the resolution is a dg-algebra, equipped with an associative multiplication that also fulfills the Leibniz rule for interacting with the differential, then there is no reason to expect the resolution to retain the algebraic structure when cutting it down in size with these methods. It turns out, however, that there will be an A_∞-structure appearing, induced by the algebra structure on the larger resolution.

In group cohomology, this is precisely what occurs. The cohomology ring is computed as the homology of a much larger dg-algebra. It happens to be associative, mainly because the situation at hand is well-behaved in itself. But the formation of homology forms a reduction of the original dg-algebra in exactly the kind of manner that induces an A_∞-structure on the results. It turns out that the information retained in this induced A_∞-structure, while not necessary for associativity of the cohomology ring, is enough to recover the information lost by computing the group cohomology: in good cases, we can reconstruct the group algebra (up to isomorphism) from the cohomology ring and its A_∞-algebra structure.

1.2.1 Accomplishments

In this thesis, we consider two specific approaches to computing A_∞-algebra structures on finite group cohomology algebras.

First off we study an algorithm for inductive construction of higher products, derived from Kadeishvili's proof of the minimality theorem (see Section 2.2.1). The main benefit of this approach is that the endomorphism ring of a resolution can be viewed as a computational black-box – we need to be able to perform computations inside $\mathrm{End}_{kG}(pk)$ and $H^*(G,k)$, but we do not necessarily need a complete description of $\mathrm{End}_{kG}(pk)$. Due to this property, we shall use the term *black-box computation* for this approach through the thesis.

The black-box computation approach has some drawbacks. The biggest is that we face infinite search spaces in several directions: an A_∞-algebra has infinitely many arities that we need to compute, and for each arity, if the vector space we build the A_∞-algebra structure on has infinitely many elements, then every tensor power of that space will also have infinitely many elements, which all would need to be checked. In Section 4.3.1 we prove a few lemmata that help us reduce the search space to a finite workload for each arity given restrictions on the cohomology ring we study. These restrictions are fulfilled for $H^*(C_n,k)$, and thus we get a more detailed computation of the results by Madsen (2002).

I have implemented this black-box algorithm in Magma. See more information on the implementation in Section 4.3.

Furthermore, application of the Künneth theorem and the Saneblidze-Umble diagonal on the associahedra allow us to compute new A_∞-algebra structures from old. We consider the specific case of computing an

A_∞-structure on $H^*(C_n \times C_m, k) \cong H^*(C_n, k) \otimes_k H^*(C_m, k)$ using the A_∞-structures on $H^*(C_n, k)$ and $H^*(C_m, k)$. We show in Section 3.3 that in the arities $2(n-2)+m$ and $2(m-2)+n$, there is an input that renders a non-zero value under the corresponding higher product.

1.3 Notation and conventions

Through this thesis, we shall use k to denote some field. The finite field with p elements is denoted by $\mathbb{F}_p$.

We expect the reader to be familiar with basics of homological algebra and group theory, but will explain concepts rooted in group cohomology or operad theory at some length.

In order to fix notation, we further declare C_n to be the cyclic group of order n, and D_n the dihedral group of order n, giving the rotation and reflection symmetry group of the $n/2$-gon. By Q_8 we shall mean the quaternionic unit group.

We write $\mathtt{Mod}_R$ for the category of (left) R-modules with all R-linear module maps. We won't consider right module categories in this thesis. We write $\mathtt{Vect}_k$ for the category of k-vector spaces with all linear maps.

All tensor products of vector spaces with additional structure are over the base field. Hence $A \otimes B$ means $A \otimes_k B$. Tensor powers are denoted by $A^{\otimes i} := A \otimes \cdots i \text{ times} \cdots \otimes A$.

A multilinear map is a map taking n arguments, linear in each argument, and returning a single value. Hence, such a map has the form $A^{\otimes i} \to B$. The degree of a basis element in a tensor product of graded vectorspaces is the sum of the degrees of the factors. Hence $|a_1 \otimes \cdots \otimes a_n| = |a_1| + \cdots + |a_n|$. The degree of a multilinear map is the drop of degree from the arguments to the value. Hence $|f| = |a_1 \otimes \cdots \otimes a_n| - |f(a_1, \ldots, a_n)|$. Differentials have degree ± 1.

A quasi-isomorphism of differential graded vector spaces is a map such that the induced map in homology is an isomorphism.

1.4 Overview

Chapter 1 gives an introduction to the thesis subject matter, fixes some notation and contains this overview of the thesis structure.

Chapter 2 introduces A_∞-algebra structures, gives overviews over several different constructions of A_∞-algebras, and states the most relevant results from the literature on A_∞-algebras and their use in representation theory. Furthermore, in Section 2.3, group cohomology rings and methods to calculate their structures are introduced. This section assumes a working knowledge of homological algebra, but no acquaintance with the study of group cohomology in particular.

Chapter 3 introduces the Saneblidze-Umble construction of a diagonal on the associahedron, and discusses its rôle in computing A_∞-structures on the cohomology of abelian finite groups. In Section 3.3, original results characterizing the A_∞-structure on $H^*(C_n \times C_m, \mathbb{F}_p)$ are discussed and proven.

Chapter 4 discusses various computational paradigms for A_∞-structures, and their relative merits and problems. In particular, in Section 4.3, we discuss the computer algebra packages written by the author to handle black-box computation of A_∞-structure maps in group cohomology, and the partial results that can be had on A_∞-structures on $H^*(D_8, \mathbb{F}_2)$ and $H^*(D_{16}, \mathbb{F}_2)$ using these programs.

I use the term *black-box computation* to denote a process for computing an A_∞-structure using an oracle that allows me to compute specific multiplications within the structure at hand instead of using a global description of the structure in question.

In Appendix A, we finally give source code and implementation discussions, as well as usability and availability discussions for all the programs developed as part of the thesis work. Specifically, we discuss implementations in Haskell of the Saneblidze-Umble in Section A.1 and a Magma implementation of black-box computation for use with p-group cohomology in Section A.2.

Chapter 2

A_∞-algebras

> To see a world in a grain of sand
> and heaven in a wild flower
> To hold infinity in the palm of your hand
> and eternity for an hour.
>
> *Auguries of innocence*
> William Blake

2.1 Definitions

In order to handle associativity up to homotopy in the context of topological H-spaces[1] and infinite loop spaces, Stasheff (1963) introduced, A_n and A_∞-spaces, which had a family of operations controlling the homotopies between the associativity relations of the multiplication and higher associativity relations of the multiplication and the introduced homotopies.

We shall work here with four different, but equivalent, definitions of an A_∞-structure.

Common for all is that an A_∞-structure is given, in a monoidal category $\mathtt{C}$, by an object A and a family of morphisms $\mu_i \in \mathrm{Hom}_{\mathtt{C}}(A^{\otimes i}, A)$ that fulfill conditions stated in various ways.

Furthermore, we choose to follow the Getzler-Jones sign convention, as is used also in the papers by Keller and Lu-Palmieri-Wu-Zhang. (Getzler and Jones, 1990; Keller, 2001; Lu, Palmieri, Wu, and Zhang, 2004; Lu et al., 2006) We shall for our discussion limit us to the class of categories $\mathtt{Mod}_R$, for R a k-algebra over a field k.

[1] topological spaces with a continuous multiplication of points, not necessarily associative

2.1.1 Stasheff axioms

The axioms formulated by Stasheff give the most explicit of the definitions available. It lists the conditions for an A_∞-structure in terms of explicit equations that should be fulfilled.

Definition 2.1.1. An A_∞-algebra over a ring R is a $\mathbb{Z}$-graded R-module

$$A = \bigoplus_n A^n$$

endowed with multilinear maps of degree $2-n$

$$m_n \colon A^{\otimes n} \to A, \quad n \geq 1$$
$$m_n \colon (A^{\otimes n})^s \to A^{s+2-n}$$

satisfying the relations given by the Stasheff identities

$$\mathrm{St}_n \colon \sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t} \circ 1^{\otimes r} \otimes m_s \otimes 1^{\otimes t} = 0 \quad .$$

We may further define an A_n-algebra as a vector space with operations $m_1, \dots, m_n$ such that the identities $\mathrm{St}_1, \dots, \mathrm{St}_n$ are all fulfilled.

These identities are viewed as functional identities, with 1 being the identity function. When bringing actual elements into the mix, we will need additional signs occuring due to the Koszul sign rule

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$

which produces signs according to the degrees of elements that commute through the tensor products.

The first few Stasheff identities have familiar interpretations.

To start with, we have $\mathrm{St}_1 \colon m_1 \circ m_1 = 0$, which together with the degree $|m_1| = 1$ means that m_1 actually works as a differential on the graded vector space A.

The next identity $\mathrm{St}_2 \colon m_2 \circ m_1 \otimes 1 + m_2 \circ 1 \otimes m_1 - m_1 \circ m_2 = 0$ is just a reformulation of $m_1 \circ m_2 = m_2 \circ m_1 \otimes 1 + m_2 \circ 1 \otimes m_1$. This equation is recognizable as the Leibniz rule, stating that with respect to the "multiplication" m_2, the map m_1 is a graded derivation and with St_1, this derivation is a differential compatible with m_2. We shall see that m_2 need not necessarily be associative, hence will not behave in all ways expected as an algebra product.

Applying the expression in the identity St_2 to elements $a, b \in A$, we get the Koszul signs acting, using $m_2 = (\cdot)$ and $m_1 = d$. Keep in mind that $|1| = 0$ and $|m_1| = 1$:

$$(-1)^{|a|\cdot|1|} m_2(m_1 a \otimes b) + (-1)^{|a|\cdot|m_1|} m_2(a \otimes m_1 b) = da \cdot b + (-1)^{|a|} a \cdot db$$

The resulting expression is familiar as the expected Leibniz rule in a graded setting.

Furthermore, the third Stasheff identity is

$$\begin{aligned} m_3 \circ 1 \otimes 1 \otimes m_1 + m_3 \circ 1 \otimes m_1 \otimes 1 + m_3 \circ m_1 \otimes 1 \otimes 1 + m_1 \circ m_3 \\ - (m_2 \circ 1 \otimes m_2 - m_2 \circ m_2 \otimes 1) = 0 \end{aligned}$$

which we can interpret as giving m_3 as a null-homotopy of the associator. Rewriting it, we get

$$m_3 \circ d_{A^{\otimes 3}} - (-1)^{|m_3|} d_A \circ m_3 = m_2 \circ 1 \otimes m_2 - m_2 \circ m_2 \otimes 1$$

where we construct $d_{A^{\otimes 3}}$ as $d_A \otimes 1 \otimes 1 + 1 \otimes d_A \otimes 1 + 1 \otimes 1 \otimes d_A$. This forms a differential on the tensor power induced by the differential on A. Among the consequences of these axioms, we find that if A is an A_∞-algebra, then A will not, in general, be associative.

Any differential graded algebra A is an A_∞-algebra with all m_i, $i \geq 3$ vanishing.

An A_∞-algebra is said to be *minimal* if $m_1 = 0$. It is said to be *strictly unital* if there is some element $1 \in A$, called the unit, such that $m_1(1) = 0$ and $m_2(1, a) = a = m_2(a, 1)$ hold and additionally $m_n(a_1, \ldots, a_n) = 0$ whenever $n > 2$ and some $a_i = 1$.

Homomorphisms

Definition 2.1.2. A *homomorphism* of A_∞-algebras from A to B is a family of R-linear maps $f_n : A^{\otimes n} \to B$, for $n \geq 1$, of degree $1 - n$ such that the Stasheff morphism identities

$$\mathrm{St}_n^m \colon \sum_{n=r+s+t} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{\substack{1 \leq r \leq n \\ n = i_1 + \cdots + i_r}} (-1)^{\sigma} m_r(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

with $\sigma = \sum_{k=1}^{r-1} k(i_{r-k} - 1)$ are fulfilled.

An A_∞-homomorphism f such that f_1 is a quasi-isomorphism of chain complexes is called a *quasi-isomorphism* of A_∞-algebras. We call f *strict* if $f_i = 0$ for all $i \neq 1$. The *identity* homomorphism is the strict homomorphism defined by $f_1 = \mathrm{Id}_A$.

The composition of two A_∞-homomorphisms f and g is defined to be given by

$$(f \circ g)_n = \sum_{\substack{1 \leq r \leq n \\ n = i_1 + \cdots + i_r}} (-1)^\sigma f_r(g_{i_1} \otimes \cdots \otimes g_{i_r})$$

with σ as above.

A homomorphism between strictly unital A_∞-algebras is said to be *strictly unital* if $f_1(1) = 1$ and $f_n(a_1, \ldots, a_n) = 0$ whenever $a_i = 1$ for some i. The strictly unital A_∞-algebra A is said to be *augmented* if there is a strictly unital morphism $\epsilon\colon A \to k$ such that the composition $\epsilon\eta = 1$, where $\eta\colon k \to A$ maps $1_k \mapsto 1_A$.

2.1.2 Free resolutions of operads

We can also define A_∞-structures using free resolutions of finitely presented operads. We start by building up our vocabulary.

Definition 2.1.3. An (non-Σ) *operad* $\mathcal{P}$ enriched in a category $\mathtt{C}$ is an indexed family $\{\mathcal{P}_n\}_{1 \leq n < \infty}$ of objects from $\mathtt{C}$ together with a family of *composition maps* $\circ_i\colon \mathcal{P}_n \otimes \mathcal{P}_m \to \mathcal{P}_{n+m-1}$ such that all compositions are associative. In other words, we demand $p \circ_i (q \circ_j r) = (p \circ_i q) \circ_{i+j-1} r$.

An alternative definition is

Definition 2.1.4. A (non-Σ) *operad* $\mathcal{P}$ enriched in a category $\mathtt{C}$ is an indexed family $\{\mathcal{P}_n\}_{1 \leq n < \infty}$ of objects from $\mathtt{C}$ together with a family of *composition maps* $\gamma\colon \mathcal{P}_n \otimes \mathcal{P}_{m_1} \otimes \cdots \otimes \mathcal{P}_{m_n} \to P_{m_1 + \cdots + m_n}$ such that γ is associative.

We write $||p|| = n$ for $p \in \mathcal{P}_n$. We call $||p||$ the *arity* of the element p.

In this thesis, we shall always work with operads in $\mathtt{Vect}_k$. Thus our operads are graded k-vector spaces with composition maps. Here we can work with *unitary* operads, in which $\dim \mathcal{P}_1 = 1$ and we can pick a basis element in $\mathcal{P}_1$ which operates as the identity. Writing $\mathcal{P}_1 = k\mathbb{1}$, we note that the two operad definitions are equivalent. Indeed,

$$\gamma(p; q_1, \ldots, q_n) = \Big((p \circ_1 q_1) \circ_{|q_1|+1} q_2 \cdots \circ_{\sum |q_i|+1} q_n\Big)$$

and in the other direction,

$$p \circ_i q = \gamma(p; \mathbb{1}, \ldots, \mathbb{1}, q, \mathbb{1}, \ldots, \mathbb{1})$$

Definition 2.1.5. If each component $\mathcal{P}_n$ in an operad $\mathcal{P}$ is a graded vectorspace, and there is some homogenous endomorphism d of degree -1 such that $d(p \circ_i q) = dp \circ_i q + (-1)^{\|p\|} p \circ_i dq$ and $d^2 = 0$, then we call $(\mathcal{P}, d)$ a *differential graded operad*.

Example 2.1.6. Consider the operad $\mathcal{E}nd(V)$ with components given by $\mathcal{E}nd(V)_n = \mathrm{Hom}_{\mathtt{Vect}_k}(V^{\otimes n}, V)$. Its elements are multilinear maps, and composition works by

$$(f \circ_i g)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+k}), a_{i+k+1}, \ldots, a_n)$$

This operad we call the *endomorphism operad* of the vector space V.

If V is a differential graded vector space, with differential d_V, then we can make $\mathcal{E}nd(V)$ a differential graded operad by introducing grading of endomorphisms by drop in degree, and the homotopy differential $\partial f = d_V f - (-1)^{|f|} f d_{V^{\otimes \|f\|}}$, where $d_{V^{\otimes a}} = \sum_{k=0}^{a-1} 1^{\otimes k} \otimes d_V \otimes 1^{\otimes a-k}$ is the usual tensor product differential.

Note that in this example, we did not really have any notable dependency on the category we were working in. Indeed, as long as $\mathtt{C}$ is a monoidal category, we can define $\mathcal{E}nd_{\mathtt{C}}(A)$ for any object A in the category. This is at the core of the algebraic uses of operads for codifying algebraic theories.

Definition 2.1.7. Let $\mathcal{P}$ be an operad in a category $\mathtt{C}$. A *representation of* $\mathcal{P}$ or a $\mathcal{P}$*-algebra* is an object $A \in \mathtt{C}$ together with a morphism of operads $\mathcal{P} \to \mathcal{E}nd_{\mathtt{C}}(A)$.

We can construct a *free operad* by using labelled rooted planar trees.

An embedding of an undirected tree with a single marked vertex yields a planar directed rooted tree by directing all edges toward the marked vertex, and calling this marked vertex the root. The indegree (outdegree) of a vertex v is the number of directed edges entering (exiting) v. We let $\mathcal{T}ree(n)$ be the set of isomorphism classes of such trees with n leaves labeled clockwise – whereby a leaf is a vertex of indegree 0. By the construction, the only vertex with outdegree different from 1 will be the root – which has outdegree 0. A tree is said to have arity the number of leaves, and a vertex is said to have arity its indegree. We shall define $\circ_i$ on the collection $\{\mathcal{T}ree(n)\}_{n \geq 1}$ to make this an operad. We do this by letting $S \circ_i T$ be the tree we get by replacing the leaf of S labeled i with the root of the tree T and then relabeling all labels according to the new tree.

Thus, the leaf in S labeled with $j < i$ retain their labels, the leaf in T with label j is assigned the new label $j + i - 1$, and the leaf in S with label $j > i$ is assigned the new label $j + \|T\| - 1$. We call this operation *grafting*.

Definition 2.1.8. The *free operad* on the graded set $X = X_1 \sqcup X_2 \sqcup \cdots$ of vertices is the operad we obtain by taking all trees with all non-leaf vertices labeled by elements of X such that the degree of the label is equal to the arity of its vertex. Composition is defined by grafting, as above, with the label on the root vertex kept intact. We write $k\langle X \rangle$ for this operad, in analogy with free algebras.

By linearity of composition, we can extend any composition of linear combinations of labeled trees to a linear combination of labeled trees. This allows us to talk about *ideals* as subsets of the free operad on some set that are closed under vector space operations and under composition with elements of the operad. Equivalence class building works as expected as well. Thus, we can consider the quotient operad of a free operad with some ideal. Again, in analogy with the notation in ring theory, we write $\langle p_1, \ldots, p_n \rangle$ for the ideal generated by the elements $p_1, \ldots, p_n$.

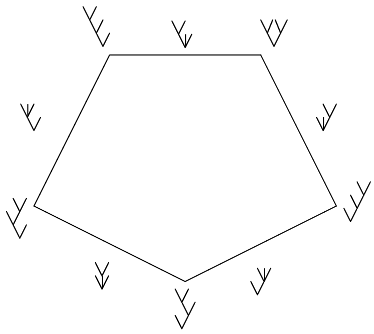
Definition 2.1.9. A *finitely presented operad* is an operad formed by a quotient of a free operad on a finite set by an ideal generated by a finite set of operad elements.

Consider now the operad controlling associative k-algebras. This is an operad over the category $\mathtt{Vect}_k$ defined as the finitely presented operad

$$\mathcal{A}ss = k\langle \vee \rangle / \langle \vee \circ_1 \vee - \vee \circ_2 \vee \rangle \quad .$$

A representation of this operad is an operad homomorphism $\mathcal{A}ss \to \mathcal{E}nd(V)$, or in other words, a family of maps $\mathcal{A}ss_n \to \mathrm{Hom}_{\mathtt{Vect}_k}(V^{\otimes n}, V)$ such that composition in the operad maps to composition of multilinear maps. Thus, a representation of the operad $\mathcal{A}ss$ is nothing further than an associative algebra: it has a binary multiplication, and all laws that can be deduced from associativity hold.

We can approach this operad with the tools familiar from homological algebra. The homology of a dg-operad $\mathcal{P}$ would be the operad with components $(H^*\mathcal{P})_n = H^*\mathcal{P}_n$, which is well defined since the differential is a differential on each component. Furthermore, since the operadic composition obeys the Leibniz rule in a dg-operad, the result is also a dg-operad. Thus, we could approximate the associativity operad by a free dg-operad $\mathcal{P}$ such that $H^*\mathcal{P} \cong \mathcal{A}ss$, where we view $\mathcal{A}ss$ as a dg-operad concentrated in degree 0. We can find this by simply constructing a free resolution step by step. We thus need to start by killing off the generating relation $\vee \circ_1 \vee - \vee \circ_2 \vee$ with a new free generator of degree 3: Ψ. This, in turn, gives rise to a higher associativity relation, as illustrated by boundary of the pentagon in the diagram below.



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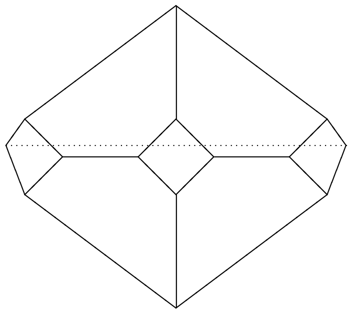
So we can introduce a new free generator of degree 4, killing this relation. However, this gives rise to another higher associativity relation, captured by the boundary of the polyhedron in the diagram below.



.

We can continue in the same manner, introducing one single generator in each degree. This generator will be the tree with one vertex and n leaves and is called a *corolla*. We equip the resulting structure with a differential d that takes a corolla to the alternating sum of all the quadratic trees of the same arity – by which we mean that dm_n is an alternating sum of trees with n leaves and with exactly two non-leaf vertices.

The result is the free dg-operad $\mathcal{A}ss_\infty$. Its homology is precisely the associativity operad. A representation of $\mathcal{A}ss_\infty$ is a differential graded vector space V and a chain map $\varrho\colon \mathcal{A}ss_\infty \to \mathcal{E}nd(V)$. For ϱ to be a chain map, we need $d\varrho = \varrho d$. Inserting the actual values everywhere, and writing $\mu_n = \varrho(m_n)$ and $d = \varrho(m_1)$, we get

$$d\mu_n \pm \mu_n d = \partial\mu_n = \sum \pm\mu_k \circ_i \mu_{n-k}$$

which by rearranging is exactly the same as St_n.

Definition 2.1.10. An A_∞ *algebra structure* on a vector space V is a dg-operad homomorphism $\mathcal{A}ss_\infty \to \mathcal{E}nd(V)$. The images of the corollas are called *higher multiplications* of the A_∞ algebra structure, and are denoted m_n.

2.1.3 Polyhedral chain maps

Yet another method to define A_∞-structures is as maps of chain complexes, starting with the complex of cellular chains on a particular polytope that carries the properties we wish to study. This approach has the benefit that some algebraic constructions we wish to perform have a highly geometric reformulation, and is easier to study by using specific realisations of the polytopes. This is leveraged in the various constructions of the diagonal on the associahedron available in the literature. See (Saneblidze and Umble, 2004; Markl and Shnider, 2006; Loday, 2007).

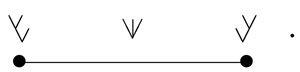
The Associahedron

Consider the set PR_n of planar trees with n leaves. We use PR_n to index the faces of a polytope, by indexing faces of dimension k by trees with $n-k-1$ internal vertices in such a way that the face indexed by T is a subface of the face indexed by T' if there is some order of collapsing internal edges of T that results in T'.

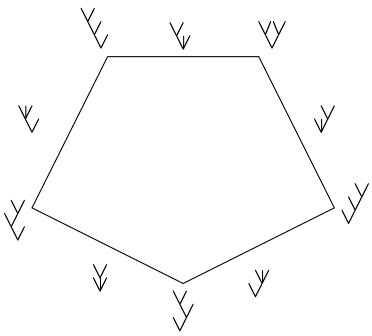
The polytope constructed in this way is called the *Stasheff polytope K_n* of dimension $n-2$, or the $n-2$-dimensional *associahedron*.

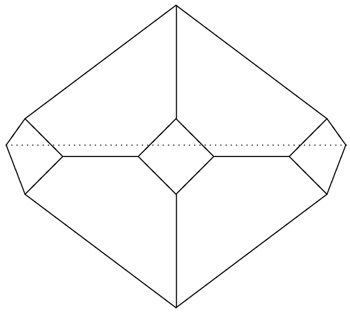
As a first example, consider the set PR_3. This consists of the trees

and we notice that the tree can be constructed from either of and by collapsing the unique internal edge. Thus, K_3 has the geometric form



Generally, the vertices of the associahedron are indexed by planar binary trees. The resulting polytopes correspond precisely to the diagrams tracking associativity conditions when introducing higher and higher associators. The next two have the shapes

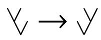


 .

There are several methods available to find geometric realizations of the associahedra. Boardman and Vogt (1973) give a way to subdivide the associahedron K_n into a union of n-cubes. Loday (2005) gives a way, using parking functions, to realize the associahedron as a simplicial complex and also a method to embed $K_n \subset \mathbb{R}^{n-2}$ with vertices at integer coordinates (Loday, 2004).

Another family of interesting polytopes are the *permutahedra*, a well-known family of polyhedra, with faces indexed by ordered partitions. One geometric realization was given by Loday (2004). Tonks (1997) gives a projection from the permutahedra onto the associahedra, which is used in the diagonal arguments in Section 3.2. For more details on Tonks construction, see Section 3.2.1.

The 1-cells of the associahedra can be given orientation ordering the vertices as a partial order, called the *Tamari order*. This ordering is induced by the basic ordering



and the covering relation in the Tamari order relates two planar binary trees precisely when we can go from one to the other by replacing a subtree of the form  with a subtree of the form .

Suppose now that R is a commutative ring.

Just as with any topological cellular complex, we can view the associahedra as cellular complexes, with cells precisely indexed by the planar trees and their adjacency controlled just as in the definition of the polytopes above. Thus, we write $C_*(K_n; R)$ for the chain complex of free R-modules with each generator indexed by a face of K_n, and thus by a planar tree with n leaves. The differential is the cellular one, with signs chosen consistent with the Stasheff axioms.

As a chain complex, this is isomorphic to the chain complex underlying the A_∞ operad defined in Section 2.1.2. Furthermore, by introducing grafting of trees as the composition and coupling this with the cellular differential, we get a dg-operad, graded by the number of internal edges, and isomorphic to $\mathcal{A}ss_\infty$.

Hence, the family of mappings

$$\mu_n \colon C_*(K_n; R) \to \mathrm{Hom}_{\mathtt{Mod}_R}(A^{\otimes n}, A)$$

factored through the $\mathcal{A}ss_\infty$-operad gives us an A_∞-structure on A with m_n the image of the top-dimensional cell of K_n under μ_n.

One large benefit of this definition is that we get a co-algebraic A_∞-structure defined for free. An A_∞-coalgebra is given by a family of chain maps

$$\theta_n : C_*(K_n) \to \mathrm{Hom}_{\mathtt{Mod}_R}(A, A^{\otimes n})$$

where the operation Δ_n is the image of the top dimensional cell of K_n under θ_n.

2.1.4 Bar construction

This section follows the exposition by Keller (2001), which in turn relies on Stasheff (1963) and Kadeishvili (1985).

Consider a graded k-vector space A. We define the *suspension* SA of A by $(SA)_n = A_{n+1}$. Thus, the suspension merely shifts grades, retaining the rest of the vector space structure. There is a canonical map $s \colon A \to SA$ of degree -1 with $sa = a$. Note that this is closely modeled after the suspension in algebraic topology – where $\Sigma X = X * \{p, q\}$ is the double cone from X to two new points. This operation shifts the entire homological structure of X one step higher in degree.

For a graded k-vector space V, we define the *tensor algebra* TV to be

$$TV = \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

where $V^{\otimes 0} = k$ and multiplication is juxtaposition. The *reduced tensor algebra* $\bar{T}V$ omits the summand $V^{\otimes 0}$.

The reduced tensor algebra $\bar{T}V$ turns into a graded coalgebra with the comultiplication defined by

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \quad .$$

Thus, $k \oplus \bar{T}V$ with Δ extended by $\Delta(1) = 1 \otimes 1$ is a graded augmented coalgebra.

A graded map $b\colon \bar{T}V \to V$ lifts uniquely to a coderivation $b\colon \bar{T}V \to \bar{T}V$, i.e. such that $\Delta b = (b \otimes 1 + 1 \otimes b)\Delta$, by setting b to $\sum_{n,m} b_n^m$ where each component map

$$b_n^m\colon T^n V \to T^m V$$

is given by the expression $\sum 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}$, where $r + s + t = n$ and $r + 1 + t = m$.

Now, consider $\bar{T}SA$. Coderivations b on $\bar{T}SA$ are in bijection with families of maps $b_n\colon (SA)^{\otimes n} \to SA$ of degree 1. We construct maps $m_n\colon A^{\otimes n} \to A$ such that the diagrams

$$\begin{tikzcd} A^{\otimes n} \arrow[r, dotted, "m_n"] \arrow[d, "s^{\otimes n}"'] & A \arrow[d, "s"] \\ (SA)^{\otimes n} \arrow[r, "b_n"'] & SA \end{tikzcd} \quad \text{commute.}$$

Then m_n has degree $-n + 1 - (-1) = 2 - n$.

Suppose now that we had a codifferential b, giving rise to maps m_n. Then, the condition $b^2 = 0$ implies, by tracking one particular component of the map b^2, that

$$T^n SA \xrightarrow{1^{\otimes j} \otimes b_k \otimes 1^{\otimes n-k-j}} T^{n-k} SA \xrightarrow{b_{n-k}} SA$$

so the condition that $b^2|_{T^n SA} = 0$ gives us that the sum

$$\sum \pm b_{n-k}(1^{\otimes j} \otimes b_k \otimes 1^{\otimes n-k-j}) = 0$$

which with an appropriate sign choice is precisely the Stasheff axiom St_n. This equation carries over to m_n by the commutative diagram.

2.2 Basic results

2.2.1 The minimality theorem

Theorem 2.2.1 (Kadeishvili (1980), see also Johansson and Lambe (2001))**.** *Let $(A, d, \cdot)$ be a dg-algebra. Then the homology chain complex H^*A has an A_∞-algebra structure such that $m_1 = 0$ and m_2 is induced by m_2^A. Furthermore, there* *is a quasi-isomorphism $H^*A \to A$ lifting the identity on H^*A. The structure is unique up to quasi-isomorphism of A_∞-algebras.*

If A has a unit inducing a unit on H^*A, then we can choose the structure to be unital and the quasi-isomorphism to be strictly unital.

Note, before we repeat Kadeishvili's proof of this theorem, that it has been proven in significantly higher generality. For more details, see the article by Johansson and Lambe (2001), or also the papers by Smirnov (1980); Gugenheim and Stasheff (1986); Gugenheim and Lambe (1989); Gugenheim, Lambe, and Stasheff (1991); Huebschmann and Kadeishvili (1991); Merkulov (1999)

Proof. This proof is identical to the one provided by Kadeishvili (1980). Some notation choices are made differently, one major difference is that Kadeishvili uses U_n where we use Ψ_n.

Let m_2 denote the induced multiplication in H^*A and choose a cocycle selection map $f_1 : H^*A \to A$.

Set $\Psi_2(a_1, a_2) = f_1(a_1 a_2) - f_1(a_1) f_1(a_2)$. This is a boundary, since $f_1(a_1 a_2)$ is defined to be a representative cycle of the homology class containing $f_1(a_1) f_1(a_2)$. Hence, there is some w such that $dw = \Psi_2(a_1, a_2)$. We define $f_2(a_1, a_2) = w$.

Now, for $n > 2$, write

$$\Psi_n(a_1, \ldots, a_n) = \sum_{s=1}^{n-1} (-1)^{\varepsilon_1(a_1,\ldots,a_n,s)} f_s(a_1, \ldots, a_s) \cdot f_{n-s}(a_{s+1}, \ldots, a_n) + \\ \sum_{j=2}^{n-1} \sum_{k=0}^{n-j} (-1)^{\varepsilon_2(a_1,\ldots,a_n,k,j)} f_{n-j+1}(a_1, \ldots, a_k, m_j(a_{k+1}, \ldots, a_{k+j}), \ldots, a_n)$$

where the expressions $\varepsilon_1(a_1, \ldots, a_n, s) = s + (n - s + 1)(|a_1| + \cdots + |a_s|)$ and $\varepsilon_2(a_1, \ldots, a_n, k, j) = k + j(n - k - j + |a_1| + \cdots + |a_k|)$ are the signs in the Stasheff morphism axiom St_n^m with the Koszul signs introduced.

This Ψ_n is the complete expression of the Stasheff morphism axiom St_n^m, but with the two terms $f_1 m_n$ and $m_1 f_n$ removed. The central point of this proof is to fill these terms back in.

By some pretty tedious technical checking, we can confirm that the element $\Psi_n(a_1, \ldots, a_n) \in \ker d$. Hence, $\Psi_n(a_1, \ldots, a_n)$ belongs to some coclass $z \in H^*A$. We define

$$m_n(a_1, \ldots, a_n) = z \quad .$$

Since now $f_1(m_n(a_1,\dots,a_n))$ and $\Psi_n(a_1,\dots,a_n)$ are in the same coclass, there is some coboundary dw, $w \in A$, such that $f_1(m_n(a_1,\dots,a_n)) - \Psi_n(a_1,\dots,a_n) = dw$. We set

$$f_n(a_1,\dots,a_n) = w \quad .$$

Since we defined everything precisely in order to match the Stasheff axioms, we end up with a structure fulfilling the Stasheff axioms.

For easier reference, we note that:

$$\begin{aligned}
\Psi_3(a,b,c) &= (-1)^{\varepsilon_1(a,b,c,1)} f_1(a)f_2(b,c) + (-1)^{\varepsilon_1(a,b,c,2)} f_2(a,b)f_1(c) + \\
&\quad (-1)^{\varepsilon_2(a,b,c,0,2)} f_2(m_2(a,b),c) + (-1)^{\varepsilon_2(a,b,c,1,2)} f_2(a,m_2(b,c)) \\
&= (-1)^{1+1\cdot|a|} f_1(a)f_2(b,c) + (-1)^{2+2\cdot(|a|+|b|)} f_2(a,b)f_1(c) + \\
&\quad (-1)^{0+2\cdot(\cdots)} f_2(m_2(a,b),c) + (-1)^{1+2\cdot(\cdots)} f_2(a,m_2(b,c)) \\
&= -(-1)^{|a|} f_1(a)f_2(b,c) + f_2(a,b)f_1(c) + f_2(m_2(a,b),c) - f_2(a,m_2(b,c))
\end{aligned}$$

As for unitality – first we consider $m_2(1,a) = m_2(a,1) = a$, and hence we can compute $\Psi_2(1,a) = a - a = 0$ and $\Psi_2(a,1) = a - a = 0$. Thus, we can safely choose $f_2(1,a) = f_2(a,1) = 0$.

Now, consider Ψ_3. We have three cases to consider:

$$\begin{aligned}
&\Psi_3(1,a,b) \\
&= -(-1)^{|1|} f_1(1)f_2(a,b) + f_2(1,a)f_1(b) + f_2(m_2(1,a),b) - f_2(1,m_2(a,b)) \\
&= -f_2(a,b) + 0 + f_2(a,b) - 0 = 0 \\
&\Psi_3(a,1,b) \\
&= -(-1)^{|a|} f_1(a)f_2(1,b) + f_2(a,1)f_1(b) + f_2(m_2(a,1),b) - f_2(a,m_2(1,b)) \\
&= -0 + 0 + f_2(a,b) - f_2(a,b) = 0 \\
&\Psi_3(a,b,1) \\
&= -(-1)^{|a|} f_1(a)f_2(b,1) + f_2(a,b)f_1(1) + f_2(m_2(a,b),1) - f_2(a,m_2(b,1)) \\
&= -0 + f_2(a,b) + 0 - f_2(a,b) = 0
\end{aligned}$$

Hence, $\Psi_3 = 0$ whenever one input is a unit, and thus $m_3 = 0$ when one input is a unit and we can choose $f_3 = 0$ when one input is a unit.

Consider now some $n > 3$. In the expression for Ψ_n, we have terms of the forms

$$\begin{aligned}
&f_i(a_1,\dots,a_i)\cdot f_j(a_{i+1},\dots,a_n) && \text{and} \\
&f_i(a_1,\dots,m_k(a_j,\dots,a_{j+k}),\dots,a_n) && .
\end{aligned}$$

In the case that $a_1 = 1$ or $a_n = 1$, the 1 occurs inside some f_k or m_k, with $k > 1$, for all cases except for the terms $(-1)^{1+(n-2)|1|} f_1(1) f_{n-1}(a_2, \dots, a_n)$ and $(-1)^{0+2\cdot(\dots)} f_{n-1}(m_2(1, a_2), \dots, a_n)$. These have opposite signs, and thus their sums vanish.

Otherwise, in the terms of the first kind, the unit occuring must occur as an argument to one of the two f_*s. Hence, that vanishes, and thus so does the entire term.

In the terms of the second kind, if the unit occurs outside the m_k, then the whole term vanishes. If the unit occurs within the m_k, then we distinguish between $k > 2$ and $k = 2$. If $k > 2$, then by assumption, m_k vanishes.

Thus, we only need to consider the case $k = 2$. In this case, we have two non-vanishing terms occuring. With signs, these are

$$(-1)^j f_i(a_1, \dots, a_{i-1}, m_2(1, a_{i+1}), \dots, a_n) = \\ (-1)^j f_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$$

$$(-1)^{j-1} f_i(a_1, \dots, m_2(a_{i-1}, 1), a_{i+1}, \dots, a_n) = \\ -(-1)^j f_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \quad .$$

Hence these two terms cancel each other, and we can conclude $\Psi_n = 0$. Thus $m_n = 0$ follows and we can safely choose $f_n = 0$.

Since we will not use the internal structure of the uniqueness proof, we refer the reader to (Kadeishvili, 1980) for that part. □

There are many choices involved in computing the A_∞-structure. However, any way we make these choices will give us a quasi-isomorphic structure, by the uniqueness in the theorem. Furthermore, Johansson and Lambe (2001) argue that three mainstream computational techniques, one of which is a generalization of Kadeishvili's methods, give not only quasi-isomorphic structures but identical structures. Thus, working with the algorithm derived from Kadeishvili's proof will give us the results desired.

This inductive method lies at the base of my black-box computation methods for A_∞-algebra structures in group cohomology. See Section 4.1 for a deeper discussion.

If A is an A_∞-algebra, we call an A_∞-structure on H^*A a *model* of A. Again, it is called a *minimal model* if $m_1 = 0$ on H^*A. The minimality theorem, in this language use, states that any dg-algebra has a minimal model. The more general versions discussed, e.g. by Johansson and Lambe (2001), are more powerful, and prove that there are minimal modles for any A_∞-algebra. An A_∞-algebra A is called *formal* if we can choose a minimal model of A in which the only non-vanishing operation is m_2.

2.2.2 The higher multiplication theorem

Keller (2002, 2001) introduced the study of A_∞-algebra structures into representation theory. At the core of the results is a close connection between $\operatorname{Ext}^*_R(\bigoplus S, \bigoplus S)$ and R, where the sum $\bigoplus S$ runs over all simple R-modules. These results were originally stated without detailed proof for quiver algebra quotients – i.e. algebras with only one-dimensional simple modules. They were proven by Segal (2007). Lu, Palmieri, Wu, and Zhang (2004, 2006) extended Kellers results to also cover abelian group graded local rings.

First off, we have an application of the minimality theorem. We set $M = \bigoplus_{S \text{ simple}} S$. We can compute $\operatorname{Ext}^*_R(M,M)$ as $H^* \operatorname{Hom}_R(pM, pM)$ with pM denoting a projective resolution of M and the differential on $\operatorname{Hom}_R(pM, pM)$ given by $df = d_{pM} f - (-1)^{|f|} f d_{pM}$, the induced differential. This gives $\operatorname{Hom}_R(pM, pM)$ a dg-algebra structure with the induced differential as differential and composition of chain maps as multiplication, so by the minimality theorem $\operatorname{Ext}^*_R(M,M)$ has an A_∞-structure, unique up to quasi-isomorphism.

Theorem 2.2.2 (Keller, Lu-Palmieri-Wu-Zhang,Segal)**.** *Suppose that R is a quiver algebra quotient or an abelian graded local ring and M is the direct sum of all simple R-modules. Then $\operatorname{Ext}^*_R(M,M)$ is generated, as an A_∞-algebra, by $\operatorname{Ext}^1_R(M,M)$.*

Theorem 2.2.3 (Higher multiplication)**.** *Suppose R and M are as in Theorem 2.2.2. Then, setting $Q = \operatorname{Ext}^1_R(M,M)$ and $I = \operatorname{Ext}^2_R(M,M)$, there is an inclusion map $\tau\colon I \to TQ$ such that $R = TQ/\tau(I)$. Furthermore, the linear dual τ^* of τ is $\tau^* = \sum_i m_i$.*

Note that this theorem means that we can recover R with an explicit presentation from an A_∞-structure defined on $\operatorname{Ext}^{\leq 2}_R(M,M)$.

2.3 Group cohomology

The main objective in this thesis is to consider the extent to which A_∞-structures can aid the study of group cohomology. Thus, an introduction to group cohomology might be of use. The results and arguments presented here can be found in more detail in any standard textbook on homological algebra – such as the books by Weibel (1994); Mac Lane (1995); Hilton and Stammbach (1997).

2.3.1 Three equivalent definitions

We define the cohomology $H^*(G,k)$ of a group G to be $\mathrm{Ext}^*_{kG}(k,k)$. We shall not in this thesis consider any other coefficients for group cohomology than the trivial module k. Furthermore, we shall expect k to be a field of characteristic dividing $|G|$.

However, $\mathrm{Ext}^*_{kG}(M,M)$ has several different and equivalent definitions. We shall display three definitions here – all of which have uses in our exposition. For the following sections, we shall develop the theory for a noetherian k-algebra R, but the applications we have in mind are all for $R = kG$. We write, throughout, k for both the base field and the trivial R-module.

Equivalence classes of cocycles

We recall that $\mathrm{Ext}^*_R(A,B)$ is defined as the right derived functors of the left exact functor $\mathrm{Hom}_R(-,B)$. If P_* is a projective resolution of A, we get a differential on $\mathrm{Hom}_R(P_*,B)$ by $df = f \circ d_{P_*}$. Thus, we can compute $\mathrm{Ext}^n_R(A,B)$ as the n$^{\text{th}}$ homology $H^n(\mathrm{Hom}_R(P_*,B))$. Thus, elements of $\mathrm{Ext}^n_R(A,B)$ are equivalence classes of maps $P_n \to B$, with pointwise module operations turning $\mathrm{Ext}^n_R(A,B)$ into an R-module, and $\mathrm{Ext}^*_R(A,B)$ into a graded R-module.

The Yoneda algebra

Recall that the set of all equivalence classes of exact sequences on the form

$$\xi : 0 \to B \to X_{n-1} \to \cdots \to X_0 \to A \to 0$$

under the equivalence relation generated by the relation that sets $\xi \sim \xi'$ if there is a commutative diagram

$$\begin{tikzcd}
\xi : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & X_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & B \arrow[r] & X'_{n-1} \arrow[r] & \dots \arrow[r] & X'_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

form an abelian group.

We recall a standard lemma:

Lemma 2.3.1. *Suppose that X_* is an exact sequence, P_* is an exact sequence of projectives and $\gamma\colon P_n \to X_0$. Then γ lifts to a chain map $\gamma_* : P_* \to X_*$ of degree n, with $\gamma_0 = \gamma$. Furthermore, any two such γ_* are chain homotopic.*

The proof utilizes projectivity of the P_i, and can be found, for instance, in Weibel (1994).

These equivalence classes of exact sequences are in bijection to coclasses in $\mathrm{Ext}^n_R(A, B)$. Consider an exact sequence ξ and a projective resolution P_* of A. Then the identity on A lifts to a diagram

$$\begin{tikzcd}
& P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "\gamma"'] & P_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : & 0 \arrow[r] & B \arrow[r] & X_{n-1} \arrow[r] & \dots \arrow[r] & X_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

and we take the equivalence class containing γ as the image of ξ in $\mathrm{Ext}^n_R(A, B)$. There is such a class since by commutativity of the leftmost square, $(d\gamma)_n = \gamma\partial_n = 0$, and thus γ really is a cocycle.

For the converse, we first observe that the submodule of cocycles in the module $\mathrm{Hom}_R(P_n, B)$ is isomorphic to $\mathrm{Hom}_R(\partial_{n-1}P_n, B)$ in a canonical manner. Recall that $\partial_{n-1}P_n \cong P_n/\ker\partial_{n-1}$. Pick γ' a cocycle in $\mathrm{Hom}_R(P_n, B)$. Since γ' is a cocycle, it vanishes on $\partial_n P_{n+1}$. By exactness, it thus vanishes on $\ker\partial_{n-1}$, and thus the map $\gamma : P_n/\ker\partial_{n-1} \to B$ induced by γ' is welldefined, and by exactness $\gamma \in \mathrm{Hom}_R(\partial_{n-1}P_n, B)$.

Now, consider some cocycle $\gamma' \in x \in \mathrm{Ext}^n_R(A, B)$ represented by $\gamma \in \mathrm{Hom}(\partial_{n-1}P_n, B)$ as above. We can choose the resolution P_* in such a way that γ is surjective. Let $L_\gamma = \ker \gamma$. Then we have a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
 & L_\gamma \arrow[r, equal] \arrow[d] & L_\gamma \arrow[d] & & & & & \\
0 \arrow[r] & \partial_{n-1}P_n \arrow[r, "\iota"] \arrow[d, "\gamma"'] & P_{n-1} \arrow[r] \arrow[d, "\pi"] & P_{n-2} \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & P_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & B \arrow[r, "\beta"'] \arrow[d] & P_{n-1}/L_\gamma \arrow[r, "\overline{\partial}"'] \arrow[d] & P_{n-2} \arrow[r] & \ldots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0 \\
 & 0 & 0 & & & & &
\end{tikzcd}
$$

with exact rows and columns by choosing β so that the left square commutes, and since $L_\gamma \subseteq \partial_{n-1}P_n = \ker \partial_{n-2}$, the map $\partial_{n-2} \colon P_{n-1} \to P_{n-2}$ induces a map $\overline{\partial} \colon P_{n-1}/L_\gamma \to P_{n-2}$. The columns are exact since they are, respectively, sequences of the form $0 \to A \to B \to B/A \to 0$ with the corresponding inclusions and projections as the functions. The upper row is exact since it is the initial portion of a projective resolution of A. And the lower row is obviously exact everywhere except possibly for at P_{n-1}/L_γ since it's built out of the same projective resolution.

Remains to prove exactness at P_{n-1}/L_γ. Now, since γ is a surjection, any element $b \in B$ lifts to some element of $k \in \partial_n P_n$. By commutativity, we have $\beta\gamma k = \pi\iota k$, and thus, by commutativity of the next square $\overline{\partial}\beta\gamma k = 0$. Thus, $\beta b \in \ker \overline{\partial}$, which shows one half of exactness. For the second part, consider some $p \in \ker \overline{\partial}$. This p has some preimage $\overline{p} \in P_{n-1}$. Thus by commutativity, $\partial_{n-2}\overline{p} = 0$. So $\overline{p} \in \ker \partial_{n-2}$ and by exactness we get $\overline{p} \in \partial_{n-1}P_n$. Thus, by commutativity, $\beta\gamma\overline{p} = p$, and we have demonstrated $p \in \beta B$.

Thus, the equivalence class of the sequence ξ in the diagram is an appropriate image of γ in the set of equivalence classes of extensions of length n.

For proof that these two constructions are inverses of each other, we first need to prove that if we generate two different $\gamma, \gamma' \colon P_n \to B$ from the same exact sequence ξ, then these belong to the same equivalence class in $\operatorname{Ext}_R^n(A, B)$. Hence, we consider the diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[d, "\gamma"'] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d] & \dots \arrow[r] & P_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & B \arrow[r] & X_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \dots \arrow[r] & X_1 \arrow[r] & A \arrow[r] & 0 \\
P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[u, "\gamma'"] & P_{n-1} \arrow[r] \arrow[u] & P_{n-2} \arrow[r] \arrow[u] & \dots \arrow[r] & P_1 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

of two lifts of the identity on A to chain maps. Now, since these maps lift the same map, they are homotopic, and so we can find maps $h_n \colon P_n \to 0$ and $h_{n-1} \colon P_{n-1} \to B$ such that $\gamma - \gamma' = \partial_{n+1} h_n - (-1)^{|\gamma|+1} h_{n-1} \partial_n$. Now, h_n is the zero map, so we are left with $\gamma - \gamma' = (-1)^{|\gamma|} h_{n-1} \partial_n = (-1)^{|\gamma|} d h_{n-1}$. Thus $\gamma - \gamma'$ is a coboundary, and hence γ and γ' belong to the same coclass.

In the other direction we need to prove that if ξ is a sequence that gives rise to a coclass $\gamma \in \operatorname{Ext}_R^n(A, B)$, and we construct a sequence η out of γ, then $\xi \sim \eta$. The situation is described by the diagram:

$$\begin{tikzcd}
\xi : 0 \arrow[r] & B \arrow[r] & X_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \dots \arrow[r] & X_1 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & \partial_n P_n \arrow[r] \arrow[u, "\gamma"'] \arrow[d, "\gamma"'] & P_{n-1} \arrow[r] \arrow[u] \arrow[d] & P_{n-2} \arrow[r] \arrow[u] \arrow[d, equal] & \dots \arrow[r] & P_1 \arrow[r] \arrow[u] \arrow[d, equal] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
\eta : 0 \arrow[r] & B \arrow[r] & P_{n-1}/L_\gamma \arrow[r] & P_{n-2} \arrow[r] & \dots \arrow[r] & P_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

In order to determine equivalence of the two sequences ξ and η, we need a sequence of chain maps through a sequence of intermediate exact sequences. However, such a sequence is portrayed in the diagram – which thus proves $\xi \sim \eta$. This concludes the argument that the Yoneda exact sequences and $\operatorname{Ext}_R^n(A, B)$ stand in bijection.

This approach allows one to construct Ext without any references to projective modules.

Chain endomorphisms

One third very fruitful way of viewing $\operatorname{Ext}_R^*(A, B)$ is by using homotopy classes of chain maps from a projective resolution P_* of A to a projective resolution Q_* of B. The idea is close to that of cocycle representation.

Given a cocycle $\gamma\colon P_n \to B$, we first lift the map to a $\gamma_0\colon P_n \to Q_0$. This lifting exists precisely because the P_i are projective – projectivity allows us to lift maps across surjections, and the map $P_0 \to B$ certainly is surjective. This then lifts, by Lemma 2.3.1, to a full chain map $P_* \to Q_*$, uniquely up to chain homotopy.

In the other direction, if we have a chain map $P_* \to Q_*$ of degree n, then we can get a map $P_n \to B$ by composing $P_n \to Q_0 \to B$. The resulting map is the double-stroked composition in the following diagram.

$$\begin{tikzcd}
& P_{n+1} \arrow[r] \arrow[dr] & P_n \arrow[r] \arrow[dr, Rightarrow] & P_{n-1} \arrow[r] & \cdots \\
Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, Rightarrow] & B \arrow[r] & 0
\end{tikzcd}$$

Consider a projective resolution pk of a semi-simple module k, with differential d. Forgetting the differential, pk is a graded module, and we write $\mathrm{End}_R(pk)$ for the graded endomorphism algebra. We can make $\mathrm{End}_R(pk)$ a dg-algebra by setting $\partial f = df - (-1)^{|f|} fd$. Cocycles with this differential are precisely the chain endomorphisms of pk, and coboundaries are precisely null-homotopic maps. Hence, $\mathrm{Ext}^*_R(k,k)$ with the chain endomorphism interpretation is precisely $H_* \mathrm{End}_R(pk)$.

2.3.2 Multiplication of coclasses

We can equip $\mathrm{Ext}^*_R(k,k)$ with an algebra structure by introducing a multiplication. This multiplication will respect the homological grading – i.e. it will induce a map $\mathrm{Ext}^n_R(k,k) \times \mathrm{Ext}^m_R(k,k) \to \mathrm{Ext}^{n+m}_R(k,k)$. There are different ways to construct this product structure, depending on which of the above treated models for Ext we choose to use. All these products are equivalent, and also in the context of modular cohomology of finite groups equivalent to the cup product structure induced from the topological definition of group cohomology as $H^*(G,k) = H^*(BG,k)$ for BG the covering space of the group G.

The product of two extensions

$$\xi : 0 \to k \to X_n \to \cdots \to X_1 \xrightarrow{x} k \to 0$$

$$\eta : 0 \to k \xrightarrow{y} Y_m \to \cdots \to Y_1 \to k \to 0$$

is defined by

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & X_n \arrow[r] & \dots \arrow[r] & X_1 \arrow[r, dotted, "yx"] \arrow[dr, "x"'] & Y_m \arrow[r] & \dots \arrow[r] & Y_1 \arrow[r] & k \arrow[r] & 0 \\
& & & & & k \arrow[u, "y"'] & & & & .
\end{tikzcd}$$

In the chain homomorphism representation, the product is simply composition of chain maps – since we can choose the same projective resolution for k in both places, and thus $\mathrm{Ext}_R^*(k,k) = H_* \mathrm{End}_R(pk)$ for pk a projective resolution of k.

The product on cocycles, finally, is best described using the chain endomorphism representation – the first coclass representative is lifted to a chain map, and the appropriate component homomorphism is composed with the second coclass representative to compute a representative for the product.

From the equality between this product and the cup product (for a proof, see Benson (1998) or Carlson, Townsley, Valeri-Elizondo, and Zhang (2003)), and graded commutativity of the cup product, the algebra structure on $H^*(G,k)$ will be a graded commutative structure.

2.3.3 Minimal resolutions

In order to compute $\mathrm{Ext}_R^*(k,k)$, we need a projective resolution of the simple module k. One particularly nice such resolution is the *minimal resolution*, which is unique up to isomorphism of chain complexes. The minimal resolution is characterised by the condition

$$\mathrm{Im}\,\partial_n \subseteq JP_{n-1}$$

where JM is the Jacobson radical of a module M. If M is simple, then $JP_n \subseteq \ker\gamma$ for all $\gamma \in \mathrm{Hom}_{kG}(P_n, M)$. Thus, the differential on the complex $\mathrm{Hom}_{kG}(P_*, M)$ vanishes, and there is a k-vector space isomorphism $\mathrm{Hom}_{kG}(P_n, M) \cong \mathrm{Ext}^n_{kG}(M,M)$.

In sufficiently nice settings – one such being when the group G has prime power order – projective modules are free, and we can find a minimal resolution with the additional property of being built out of free kG-modules. For such resolutions, we'd start the resolution by picking a k-basis for kG consisting of $\{1\} \cup \{g-1 : g \in G\}$. The projection $\epsilon\colon kG \to k$ defined by $\epsilon(g) = 1$ for all $g \in G$ is called the *augmentation*, and it is easily seen that $\{g-1 : g \in G\}$ forms a basis for $\ker\epsilon$. We call this ideal the *augmentation ideal* and note that since $kG/\ker\epsilon = k$, the augmentation ideal is maximal in kG.

2.3.4 A_∞-structures on Ext-algebras

According to Section 2.3.1, we can consider the dg-algebra $A = \mathrm{End}_R(pk)$ with the property that $H^*A = \mathrm{Ext}^*_R(k,k)$. Hence, by the minimality theorem there is an A_∞-structure on $\mathrm{Ext}^*_R(k,k)$, as well as a quasi-isomorphism of A_∞-algebras $\mathrm{Ext}^*_R(k,k) \to \mathrm{End}_R(pk)$. Furthermore, if R has only one-dimensional simple modules – as is the case for $R = kG$ if G is a p-group – then the conditions for Keller's higher multiplication theorem apply, and the A_∞-structure on $H^*(G,k)$ determines the group ring kG completely.

2.3.5 Products of groups

In order to consider the cohomology of a direct product $G \times H$, we need first to recall one of the standard theorems of homological algebra. We shall state the theorem here for the case we will be using it for – but would like to point out that it holds in higher generality. See the exposition by Weibel (1994) for more details.

Theorem 2.3.2 (Künneth). *Suppose k is a field or a Dedekind domain. Suppose further that A and B are k-free chain complexes. Then $A \otimes_k B$ is a double complex with the differentials*

$$d'(a \otimes b) = da \otimes b \qquad d''(a \otimes b) = (-1)^{|a|} a \otimes db$$

and $d' + d''$ forms a differential for the total complex.

Then there is an exact sequence

$$0 \to H_*(A) \otimes_k H_*(B) \to H_*(A \otimes B) \to \mathrm{Tor}_1^k(H_*(A), H_*(B)) \to 0 \quad .$$

Let G and H be groups and let k be a field. Pick X_* and Y_* to be free resolutions of the simple module k over kG and kH respectively. We consider the complexes $X'_* = \mathrm{Hom}(X_*, k)$ and $Y'_* = \mathrm{Hom}(Y_*, k)$. We see $H_*(X'_*) = H^*(G,k)$ and $H_*(Y'_*) = H^*(H,k)$. Now, since kG and kH are both k-free, all modules in this discussion are k-free.

We note that $k[G \times H] \cong kG \otimes_k kH$ and that the total complex $X_* \otimes Y_*$ is a k-free resolution of k over $k[G \times H]$. The Künneth short exact sequence thus, in this case, is

$$0 \to H^*(G,k) \otimes H^*(H,k) \to H^*(G \times H, k) \to \mathrm{Tor}_1^k(H^*(G,k), H^*(H,k)) \to 0$$

where the Tor-term vanishes, since all modules in the arguments are free over k. Thus, we end up with the exact sequence

$$0 \to H^*(G,k) \otimes H^*(H,k) \to H^*(G \times H, k) \to 0$$

which demonstrates an isomorphism $H^*(G,k) \otimes H^*(H,k) \cong H^*(G \times H,k)$.

We will here also note that if G is abelian, then kG is an abelian graded ring concentrated in degrees $\{0\} \times G$ in the notation of Lu et al. (2006). Thus, the higher multiplication theorems apply to kG.

Chapter 3

Diagonals on the associahedron

> Facilius per partes in cognitionem totius adducimur
>
> Seneca

3.1 Tensor products of A_∞-algebras

A commonly occurring theme in algebra is to construct new structures from older and thus breaking down problems into smaller components. When studying and computing A_∞-structures in group cohomology, we'd be well helped if we can divide the computation problem into subproblems in some manner.

As discussed in Section 2.3.5, there is an isomorphism of k-vector spaces $H^*(G \times H, k) \cong H^*(G, k) \otimes_k H^*(H, k)$.

This begs the question: if we do know A_∞-structures on R and S, can we find an induced A_∞-structure on $R \otimes_k S$? This question boils down to whether there is an operad homomorphism $\varrho_\Delta \colon \mathcal{A}ss_\infty \to \mathcal{E}nd(R) \otimes \mathcal{E}nd(S)$.

This problem has been treated by Saneblidze and Umble (2004) and later by Markl and Shnider (2006). Furthermore, there is a simplicial construction by Loday (2007). It has been conjectured by Loday (2007), that all these constructions give the same result.

Following Saneblidze and Umble, we shall construct ϱ_Δ by composing the isomorphism $\mathcal{A}ss_\infty \to C_*(K_n)$ with a homomorphism $\Delta \colon C_*(K_n) \to C_*(K_n) \otimes C_*(K_n)$, yielding

$$\mathcal{A}ss_\infty \xrightarrow{\cong} C_*(K_n) \xrightarrow{\Delta} C_*(K_n) \otimes C_*(K_n) \xrightarrow{\theta_R \otimes \theta_S} \mathcal{E}nd(R) \otimes \mathcal{E}nd(S).$$

3.2 The Saneblidze-Umble diagonal construction

Since the operad $\mathcal{A}ss_\infty$ has arity components $\mathcal{A}ss_\infty(n) = C_*(K_n)$, the operad homomorphism Δ could be constructed from a chain complex diagonal $\Delta' : C_*(K_n) \to C_*(K_n) \otimes_k C_*(K_n)$. Then, from the tensor product $C_*(K_n) \otimes_k C_*(K_n)$ we would get to $\mathcal{E}nd(A) \otimes \mathcal{E}nd(B)$ by composing with $\theta_A \otimes \theta_B$.

The construction given by Saneblidze and Umble uses a different set of polytopes – the permutahedra, on which they can describe a diagonal map on the chain complexes. The connection back to the associahedral diagonal comes from a projection, given by Tonks, from the permutahedra to the associahedra, inducing a corresponding projection between the chain complexes.

Hence, if we can somehow pick a tree indexing a face of an associahedron, and associate to it a face on the permutahedron, then we can map this with a diagonal on the permutahedron to something in the tensor square of the chain complex of that permutahedron, and then project each factor back down to the associahedron using the projection from Tonks.

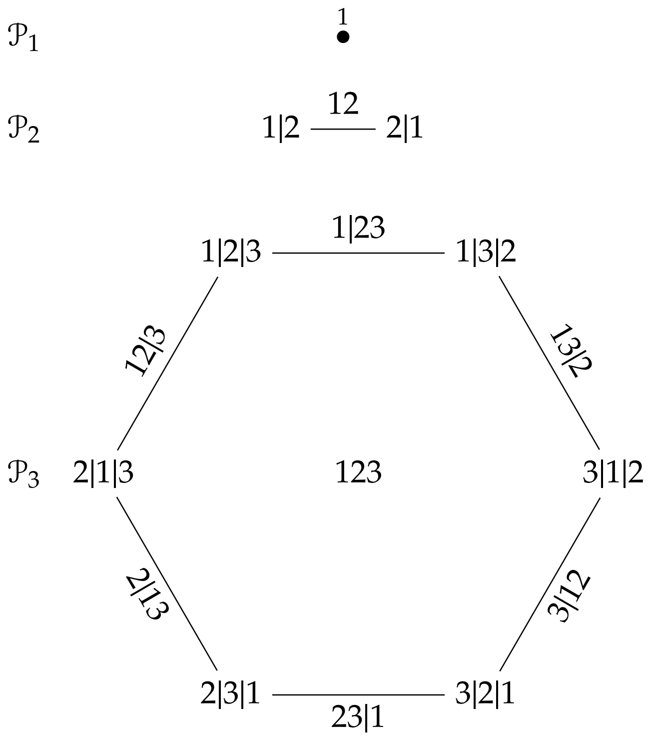
3.2.1 The permutahedron and the Tonks projection

The permutahedra are a sequence $\mathcal{P}_n$ of polytopes, similar to the associahedra, but indexed by a different set of combinatorial entities. Their name derives from that the vertices are indexed by permutations of n elements. Higher faces are indexed by *ordered partitions*:

Definition 3.2.1. An *ordered partition* of the set $[n] = \{1 \dots n\}$ is an ordered sequence $U_1|\dots|U_r$ of non-empty subsets $U_i \subseteq [n]$ such that $\bigcup_i U_i = [n]$ and $U_i \cap U_j = \emptyset$ for any $i, j \in [r]$.

Definition 3.2.2. The permutahedron $\mathcal{P}_n$ is the cellular $n-1$-dimensional polytope whose k-dimensional faces are indexed by ordered partitions $U_1|\dots|U_{n-k}$ of $[n]$. Thus, the vertices are indexed by S_n. A face $U = U_1|\dots|U_r$ is a subface of a face $V = V_1|\dots|V_s$ if U is a subdivision of the partition V.

The first few permutahedra are given by the following pictures:



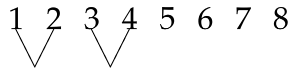
The associahedron K_{n+1} is a quotient of $\mathcal{P}_n$ by a construction by Tonks (1997). We recall that the faces of the associahedra are indexed by planar rooted trees (see Section 2.1.3). Complementary to this, we define the set PL_n of *planar leveled trees* as planar rooted trees with each internal node equipped with an integer, the *level*, in such a manner that as the tree is traversed toward the root, the integers assigned decrease. We can assign a tree from PL_{n+1} to each ordered partition of $[n]$ as follows. First, we number the leaves $1, \dots, n+1$. Then, we work through the parts of the partition in order. If i occurs in U_k, then this means that at the kth level, the leaf i gets connected with the leaf $i+1$.

Forgetting the levels, we get a map $PL_n \to PR_n$, which gives the association from partitions to associahedral faces.

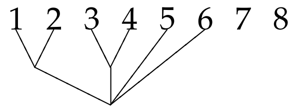
To illustrate, a few examples.

Consider the partition 13|245|67 of $[7]$. We work through the parts one by one, assembling the corresponding tree in steps:

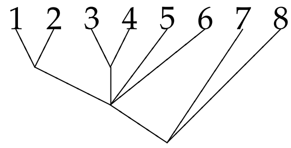
13|245|67



13|**245**|67

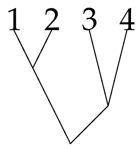


13|245|**67**



To complete the bijection, we shall describe the inverse to this map. Given a tree with $n + 1$ vertices, we construct an ordered partition on $[n]$ by considering the layers of internal nodes from the top and downwards. For each layer, consider the subtrees connected by the nodes in that layer. Each subtree has a maximal leaf. Take the labels of the maximal leaves of all subtrees except the rightmost connecting to each node. This set forms the part corresponding to that level in the ordered partition assigned to that tree.

As an example for this, consider the tree



.

This tree has three layers of nodes. We shall consider each layer in turn and construct an ordered partition U. The top layer connects the subtrees consisting of 1 and 2, respectively. So, the maximal leaves are 1 and 2, and we disregard the maximal leaf connected to each node. Thus, $U_1 = \{1\}$.

As for the second layer, this connects 3 and 4, and by the same reasoning receives the part $U_2 = \{3\}$.

Finally, the third layer connects the subtree 12 with the subtree 34. The maximal leaves are 2 and 4, and we discard 4. This finalizes the partition computation, and we get $U = 1|3|2$.

Note that we paid attention to level heights in these trees - so 1|3|2 indexes a different tree than 3|1|2 does. However, the planar trees indexing the faces of the associahedron do not take this into account. In fact, the entire edge 13|2 gets mapped to the same vertex in K_4. For these cases

– when the dimension of a face drops under the projection – we won't want to take the image into consideration at all, since the dimensions of anything we build with it will not be as expected.

This dimension dropping phenomenon is easily recognized, by looking for *derived consecutive* partitions. We define a sequence of elements $a_1, \dots, a_r$ occuring in some part U_k of an ordered partition U *derived consecutive* if all gaps in the sequence $a_1, \dots, a_r$ occur in parts U_i with $i < k$ of the ordered partition U. We call an ordered partition U *derived consecutive* if every part is derived consecutive, that is for $1 \leq j \leq r$, we require $[\min U_j, \max U_j] \subseteq \bigcup_{i \leq j} U_i$. If an ordered partition is not derived consecutive we will call it *derived non-consecutive*.

The Tonks' projection works by picking out the trees corresponding to ordered partitions as long as the partitions are derived consecutive, and by sending any derived non-consecutive partitions to $0 \in C_*(K_{n+1})$. This establishes, after expanding multiplicatively, the required projection map $C_*(\mathcal{P}_n) \to C_*(K_{n+1})$.

As an example, consider the trees indexed by 13|2, 1|3|2, 3|1|2 as considered above. These trees can be considered as three different levelings of the same tree. The form 13|2 has both top nodes in the same level, whereas 1|3|2 and 3|1|2 are the two non-equal levelings possible for the same tree. The partition 13|2 is therefore derived non-consecutive, while 1|3|2 and 3|1|2 both are derived consecutive. Under the projection, all these get sent to the same tree . However, this is a vertex on the associahedron, and 13|2 is an edge on the permutahedron, which gives a drop in dimension over the projection. The derived consecutivity filters out precisely the partitions that drop in dimension under the Tonks projection.

We shall construct a diagonal Δ_P on the cellular chain complex on the permutahedron $C_*(\mathcal{P}_n)$ by constructing matrices indexing basis elements of $C_*(\mathcal{P}_n) \otimes C_*(\mathcal{P}_n)$. Each such matrix indexes a basis element $U \otimes V$ of the tensor product $C_*(P_n) \otimes C_*(P_n)$, with one partition given by reading the columns of the matrix from left to right and one by reading the rows from the bottom and upwards. The set of all matrices acquired through the construction method thus enumerates terms in the image of the top dimensional cell of $C_*(P_n)$ in $C_*(P_n) \otimes C_*(P_n)$. Expanding multiplicatively, we get a full diagonal, and projecting onto the associahedron, we get the required associahedral diagonal.

Definition 3.2.3. A $p \times q$ matrix is called *ordered*, if the non-zero entries in each column increase downwards, and in each row increase rightwards.

A *$p \times q$-step matrix* is a $p \times q$-matrix with each of the integers $1, \dots, p + q - 1$ occuring exactly once, and all other entries carrying a 0, fulfilling

1. Non-zero entries in each column occur consecutively
2. Non-zero entries in each row occur consecutively
3. Each diagonal parallell to the main diagonal contains one single entry.

We note that step matrices are ordered by their definition.

The matrices that we get this way have entries snaking their way from the bottom left to the top right along some connected path constructed using only right- and upwards steps. The step matrices with n non-zero entries are in bijection to S_n by the following method.

Given a permutation $\pi \in S_n$, read the permutation from beginning to end. As long as the entries fall, distribute the falling sequence upwards. When the entries rise, distribute them to the right. Once the permutation is read through, we have a step matrix spanned by the entries.

Conversely, if we read a step matrix along the traced path from the bottom left to the top right, we get a permutation. This forms the inverse to the described method of constructing a matrix from a permutation, thus displaying the bijectivity.

Next, we define two matrix transformations.

Definition 3.2.4. Given an ordered matrix $M = (m_{i,j})$, we define

The right-shift $R_S M$ for $S \subset \{m_{k,j} : k \geq 1\}$ by interchanging all $m_{i,j} \in S$ with $m_{i,j+1}$ if $\min S > \max\{m_{k,j+1} : k \geq 1\}$. If $S = \{x\}$ is a singleton, we write R_x for R_S.

The down-shift $D_T M$ for $T \subset \{m_{i,k} : k \geq 1\}$ by interchanging all $m_{i,j} \in S$ with $m_{i+1,j}$ if $\min S > \max\{m_{i+1,k} : k \geq 1\}$. If $T = \{x\}$ is a singleton, we write D_x for D_T.

The elements interchanged will always be a 0 interchanged with a non-zero element, since the ordering of the matrix otherwise would prevent the conditions $\min S > \max\{m_{k,j+1} : k \geq 1\}$ and $\min T > \max\{m_{i+1,k} : k \geq 1\}$. These conditions guarantee that $R_S M$ and $D_T M$ will be ordered matrices as well.

Now, a *derived matrix* is a matrix that we construct from a step matrix by subsequent application of right shifts on the columns $C_1 \ldots C_p$ and then down shifts on the rows $R_1 \ldots R_q$, in order.

The set of all derived matrices represent terms of the diagonal. The image of a top-dimensional cell of $\mathcal{P}_n$ under the diagonal is a linear combination of terms $u \otimes v$ indexed by all matrices derived from step matrices on $[n]$. This is extended to a map from all of $C_*(P_n)$ by extending linearly over parts of the partition being mapped. For characteristics other than 2, each matrix receives a coefficient from $\{-1, 1\}$ in the linear combination, computed by computing a sign for the matrix. The sign as such is computed by using a rule for computing signs of step matrices and a rule for computing the change in sign related to right and down shifts.

More specifically, we first assign a sign to a step matrix represented by the permutation $\pi \in S_{n+1}$ as follows: We write $\mathrm{sgn}_p(\pi)$ for the usual sign of the permutation π.

Suppose that the step matrix M_π has rows $m_1 \dots m_r$. Then we define the row sign sgn_r by

$$\mathrm{sgn}_r(\pi) = (-1)^{\sum_{i=1}^{r-1} i|m_i|} \quad .$$

Finally, we define the order-reversing sign sgn_o by

$$\mathrm{sgn}_o(\pi) = (-1)^{\frac{1}{2}(\sum |U_i|^2-(n+1))} \quad .$$

With these definitions in place, we define the sign of a step matrix indexed by $\pi \in S_{n+1}$, following the definitions in (Saneblidze and Umble, 2004), to be the product

$$\mathrm{sgn}(\pi) = (-1)^{\binom{r}{2}} \mathrm{sgn}_p(\pi)\, \mathrm{sgn}_r(\pi)\, \mathrm{sgn}_o(\pi) \quad .$$

With this in place, we can define a sign for every derived matrix by giving a sign change rule that accompanies the down and right shifts. Suppose that the current face is represented by the tensor product of ordered partitions $\mu \otimes \lambda$. We define the *upper* and *lower cuts* by $(a, S] = \{s \in S | a < s\}$ and $[S, a) = \{s \in S | s < a\}$ respectively and set

$$\mathrm{sgn}(R_{i,x}(\mu \otimes \lambda)) = -\,\mathrm{sgn}(\mu \otimes \lambda) \cdot (-1)^{|(x,\mu_i] \cup [\mu_{i+1},x)|}$$

and symmetrically for downshifts

$$\mathrm{sgn}(D_{j,x}(\mu \otimes \lambda)) = -\,\mathrm{sgn}(\mu \otimes \lambda) \cdot (-1)^{|(x,\lambda_j] \cup [\lambda_{j+1},x)|} \quad .$$

We define a diagonal Δ_P on the top dimensional faces $p_n \in \mathcal{P}_n$ by

$$\Delta_P(p_n) = \sum_{p \otimes q} \mathrm{sgn}(p \otimes q) p \otimes q \tag{3.1}$$

where the sum ranges over all $p \otimes q$ indexing derived matrices, and extend it linearly to a diagonal map $\Delta_P \colon C_*(\mathcal{P}_n) \to C_*(\mathcal{P}_n) \otimes C_*(\mathcal{P}_n)$.

We can extend this to a diagonal Δ_A on the associahedron K_n by defining it on corollas k_n, by

$$\Delta_A(k_n) = \sum_{p \otimes q} \operatorname{sgn}(p \otimes q) p \otimes q \tag{3.2}$$

where the sum ranges over all $p \otimes q$ indexing *derived consecutive* matrices derived from step matrices on $[n-1]$.

3.2.2 Saneblidze-Umble diagonal term enumeration

In Appendix A.1, we give a Haskell implementation of the Saneblidze-Umble diagonal. It takes an integer parameter n and returns the full decomposition of m_n on a tensor product given either as pairs of ordered partitions or as matrices.

There are previous implementations of this particular task. Weaver (2005) presents an implementation in C++ which breaks down due to system limitations whilst listing m_6, and Tonks has private implementations in Perl and Maple, with which he has been able to enumerate the terms up to and including m_7. Neither of these implementations are widely disseminated.

My implementation can be downloaded at my university website at `http://www.minet.uni-jena.de/~mik/SaneblidzeUmble.tar.gz`. It is ready to be used with the Glasgow Haskell Compiler (University of Glasgow, 1999). Using the program `lhs2TeX` and a LaTeX-system, it can also be compiled into a self-documenting paper on the implementation.

We have tested the code and its performance by calculating, subsequently, the diagonals on $P_1, \ldots, P_8$, generating expressions for $m_2, \ldots, m_9$ on a tensor product of A_∞-algebras. The results of our calculations as well as some complexity measurements can be found in Table 3.1. The tests were performed on a double Dual Core AMD Opteron(tm) Processor 270 with 16G RAM running OpenSuSE 10.2 with a standard Linux kernel version 2.6.18.

Given the garbage collection that the Glasgow Haskell Compiler uses, there is a difference to be observed between the total amount of memory ever allocated, and the maximal amount of memory allocated at a single point in time. The measurements will state both.

n	$\lvert\Delta_{P_n}(e^n)\rvert$	Execution time	Total allocation	Peak allocation
1	1	<0.005s	45.813k	28.617k
2	2	<0.005s	53.438k	28.617k
3	8	<0.005s	95.648k	28.617k
4	50	<0.005s	402.500k	28.617k
5	432	0.02s	3.987M	56.695k
6	4 802	1.63s	45.687M	1.631M
7	65 536	399.97s	1.000G	22.198M
8	1 062 882	93 965.64s	39.205G	342.704M

Table 3.1: Performance and calculations

3.3 Computing A_∞-structures for abelian group cohomology

An A_∞-structure of $H^*(C_n, k)$ was computed by Madsen (2002). The computation is repeated in more detail in Section 4.3.3. We recall here that the structure computed has two non-trivial arities, m_2 and m_n, and that the 2-ary product of two odd coclasses vanishes while only the n-ary product of n odd coclasses does not vanish. Using this structure, the problem of computing an A_∞-structure on $H^*(G, k)$ for finite abelian groups G reduces with Künneth to the problem of computing repeated tensor products of A_∞-algebras. The computation of tensor products of A_∞-algebras, in turn, is exactly what the Saneblidze-Umble diagonal does.

An easy example should be to try and compute an A_∞-structure on $H^*(C_n \times C_m, k)$. This amounts to taking one single tensor product of two A_∞-algebras. The relative simplicity of this situation lets us state several partial results compounding to a description of the tensor product structure.

For the special case $n = m$, the dual case of computing an A_∞-coalgebra structure on the group homology has been treated by Ainhoa Berciano in studies of additional algebraic structures on tensor factors of the homology $H_*(K(\mathbb{Z}; n), \mathbb{F}_p)$.

Theorem 3.3.1 (Berciano (2006)). *The A_∞-coalgebra structure on $H_*(C_q \times C_q, \mathbb{F}_p)$ has all higher operations trivial unless the arity is one of $2, q, 2q-2$ for some k.*

The relation between Berciano's result and my own studies follows by the following lemma

Lemma 3.3.2. *Suppose that R is a k-algebra such that each $\mathrm{Ext}^n_R(k,k)$ and each $\mathrm{Tor}^R_n(k,k)$ is finite-dimensional over k. Then the following assertions hold:*

If we have an A_∞-algebra structure on $\mathrm{Ext}^*_R(k,k)$, then this gives rise to an A_∞-coalgebra structure on $\mathrm{Tor}^R_*(k,k)$.

If we have an A_∞-coalgebra structure on $\mathrm{Tor}^R_*(k,k)$, then this gives rise to an A_∞-algebra structure on $\mathrm{Ext}^*_R(k,k)$.

Proof. The universal coefficient theorem yields an isomorphism of graded k-vector spaces $\mathrm{Ext}^*_R(k,k) \cong \mathrm{Hom}_k(\mathrm{Tor}^R_*(k,k),k)$. See the expositions in Weibel (1994) or Evens (1991) for details. Write E for $\mathrm{Ext}^*_R(k,k)$ and T for $\mathrm{Tor}^R_*(k,k)$.

Suppose we have an A_∞-coalgebra structure on T. Then we have a family of $\Delta_n : T \to T^{\otimes n}$. Pick out a single element in this family. We get the diagram

$$\Delta_n : T \longrightarrow T^{\otimes n}$$

and dualizing we get

$$\mathrm{Hom}_k(T,k) \longleftarrow \mathrm{Hom}_k(T^{\otimes n},k) = \mathrm{Hom}_k(T,k)^{\otimes n} \quad .$$

Using the vector space isomorphism, this diagram gives us k-linear maps

$$(\Delta_n)^* : E^{\otimes n} \longrightarrow E$$

and this correspondence provides us with an isomorphism of k-vector spaces $\mathrm{Hom}_k(E^{\otimes n},E) \cong \mathrm{Hom}_k(T,T^{\otimes n})$. Furthermore, the isomorphism respects the operadic structure on $\mathcal{E}nd(E)$ and $\mathcal{E}nd^{\mathrm{op}}(T)$

The correspondences of A_∞-algebra structures on $\mathrm{Ext}^*_R(k,k)$ and A_∞-coalgebra structures on $\mathrm{Tor}^R_*(k,k)$ now follow by composing this isomorphism with the $\mathcal{A}ss_\infty$ representation maps. $\square$

In a slightly more general setting than Berciano's results, we can prove the following result:

Theorem 3.3.3. *Suppose that $p|n$, $p|m$ and $n,m \geq 4$. Then the A_∞-algebra structure on $H^*(C_m \times C_n, \mathbb{F}_p)$ has non-trivial operations of the arities $2, n, m, n+m-2, 2(n-2)+m, 2(m-2)+n$.*

This result, when specialized to the case $n = m$, demonstrates non-trivial operations of the arities $3(n-2)+2$. This is in stark contrast to the results Ainhoa Berciano has received, and is an issue calling for further research.

The original version of the following arguments attempted to prove the existence of non-trivial operations in $H^*(C_n \times C_m, \mathbb{F}_p)$ for all the arities on the form $k(n-2) + k(m-2) + 2$, and more, but the proof fails at a central point. After the argument, repeated here mainly for the combinatorics of the lemmata leading up to the claim, I have included a discussion of the problematic spots after stating the erroneous proof.

The argument was published in (Vejdemo Johansson, 2008), and an erratum with the discussion following the proof is submitted to the journal.

Lemma 3.3.4. *Each column of a derived matrix divides into derived consecutive blocks whose lengths index the orders of the corollas that will appear in that level.*

Proof. Suppose $a_1, \dots, a_m$ are derived consecutive in row or column j. Then the levels preceeding j in the graph will have already connected all $a_i + 1, \dots, a_{i+1}$, for all the elements failing to appear in the sequence $a_1, \dots, a_m$. Thus, in order for all a_i to meet $a_i + 1$ at the level j, all the subtrees already connecting all the gaps have to meet in one single corolla. Thus, the derived consecutive block indexes a single corolla of arity $m + 1$. □

Lemma 3.3.5. *If one factor of a term of the diagonal is constructed using only m_2, then the other factor has to be a single corolla of the appropriate arity.*

Proof. The proof is symmetric for the two possible locations for the factors, so we shall consider the case where the left factor has all m_2. This is given by the matrix

$$\begin{pmatrix} 1 & 2 & \dots & n-1 \end{pmatrix}$$

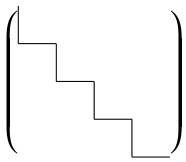
which has a single row which is a derived consecutive block in its own right, proving the claim. □

Lemma 3.3.6. *There are diagonal terms of arity $k(n-2) + k(m-2) + 2$.*

Proof. The case for $ka = 0$ is taken care of by the matrix

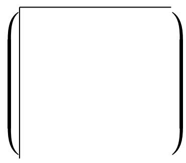
$$\begin{pmatrix} 1 \end{pmatrix}$$

There is a derived matrix of the form



(3.3)

where the picture is taken to depict a sparse matrix with non-zero entries only along the polygonal path, each horizontal line corresponding to $n-1$ consecutive integers and each vertical line corresponding to $m-1$ consecutive integers. This matrix exists since it can be constructed from a $k(m-2)+1 \times k(n-2)+1$-matrix of the form



(3.4)

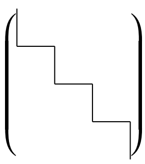
where again the polygonal path depicts the only positions in the matrix with non-zero entries. The sequence of moves constructing the matrix (3.3) from the matrix (3.4) would use right shifts and down shifts that places each block in the zigzag where it belongs. The column in this step matrix would be a sequence of blocks of subsequent integers, each block of length $n-1$ and each block ending with an element on the form $k(n-2)+(k-1)(m-2)+1$. The row would start with 1 in the first column, and then have a sequence of blocks of subsequent integers, each of length $m-1$, and each ending with an element on the form $k(n-2)+k(m-2)+1$.

This matrix can be transformed into the snake like matrix given earlier by moving each block down or right to the expected position using down shifts and right shifts. Since any element that gets moved will move past only elements that are smaller than itself, and that have stopped higher up, and higher to the left, all moves needed are admissible.

All in all, if we have k blocks down and k blocks to the right, the last element is $k(n-2)+k(m-2)+1$. Hence, the thus described operation has arity $k(n-2)+k(m-2)+2$. □

Lemma 3.3.7. *There are diagonal terms of arity $k(n-2)+(k-1)(m-2)+2$*

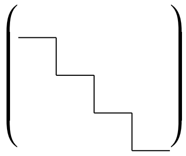
Proof. Similarily to in Lemma 3.3.6, we can construct a derived matrix on the form



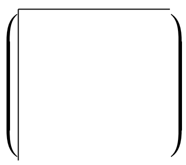
by simply dropping the last block in the top row, and proceeding with everything else just as in the proof of Lemma 3.3.6. The result has highest element $k(n-2)+(k-1)(m-2)+1$, and so the corresponding operation has arity $k(n-2)+(k-1)(m-2)+2$. □

Lemma 3.3.8. *There are diagonal terms of arity $(k-1)(n-2)+k(m-2)+2$.*

Proof. Again, similar to Lemma 3.3.6, we can construct a derived matrix on the form



which results from down shifts and right shifts from a matrix on the form



where the first row is a sequence of blocks of subsequent integers, each block of length $m-1$, and each block ending with an entry on the form $(k-1)(n-2)+k(m-2)+1$, and the first column has a 1 in the first row, and thereafter is a sequence of blocks, each of length $n-1$, and each ending with an entry on the form $k(n-2)+k(m-2)+1$.

This matrix has highest entry $(a-1)(n-2)+a(m-2)+1$, and so the corresponding operation has arity $(a-1)(n-2)+a(m-2)+2$. □

The following result, and the accompanying proof were published in Vejdemo Johansson (2008). However, there is a fundamental flaw in the argument. After stating the proof as published, we shall see counterexamples and a way to recover the statement for the arities $2(n-2)+m$ and $2(m-2)+n$.

Lemma 3.3.9. *The "snake-like" operations displayed above do not vanish as operations on $H^*(C_n \times C_m, \mathbb{F}_2)$.*

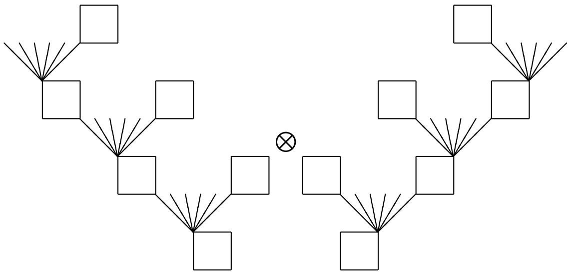
Attempted proof. We shall prove the statement for the snake-like operation of arity $k(n-2)+k(m-2)+2$. The other two cases follow by removing runs of $1\otimes x$ or $x\otimes 1$ from the proposed argument, and in the term diagram by adding boxes to the left of the uppermost corolla on the left hand side or to the right of the uppermost corolla on the right hand side.

First off, $H^*(C_n \times C_m, \mathbb{F}_2)$ has algebra generators $x\otimes 1$ and $1\otimes x$ of degree 1 and $y\otimes 1$ and $1\otimes y$ of degree 2.

Now, we consider the input that, after deshuffling may be written as

$$x|\overset{n}{\ldots}|1\overset{m-4}{\ldots}|1|x|\overset{n}{\ldots}|x\otimes 1|\overset{n-2}{\ldots}|1|x|\overset{m}{\ldots}|x|1|\overset{n-2}{\ldots}|1$$

For a diagonal term not to vanish with this argument, it will need to have the form



with the boxes consisting of trees built out of m_2's, and the tree above and below each higher corolla containing, together, 2 less inputs than the corollas on the other side of the tensor product, since the running blocks of x's need to hit the larger corollas, and the 1's cannot hit the larger corollas, lest the term vanishes.

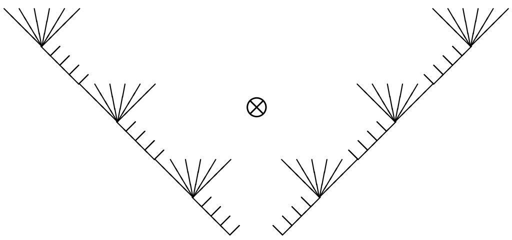
Thus, by considering the structure of the left hand tree, the first column must contain $1, 2, \ldots, n-2, k$, where k is one more than the highest occuring digit in the first box. Thus, in order for the term not to vanish under the Tonks projection, we need $k = n-1$.

Continuing down the tree, we get, since $k = n-1$, that after the column with $1, 2, \ldots, n-1$, we get a sequence of columns containing one digit each, ending with $n-1+m-2$. Then, $(n-2)+(m-2)+1, \ldots, (n-2)+(m-2)+(n-2)$ have to occur in a single column, to accomodate the next corolla, and again, in order for the term not to vanish under the Tonks' projection, we cannot have anything in the box above and to the right of the corolla.

We can continue this argument to conclude that on the left hand side, all the upper right boxes actually vanish.

By symmetry, and by repeating the argument for the right hand tree from the bottom up, we show that all the upper left boxes vanish.

Thus, any tree that does not vanish on the given arguments has the form



.

This corresponds precisely to the snake-like term, and no other term, of the diagonal, which shows that for this particular set of arguments we do get a non-vanishing value. □

The proof fails due to the assertion of the snake-like form being the only one not vanishing on the given input. Indeed, for $k = 2$, the input for an operation of arity $2(n-2) + 2(m-2) + 2$ will additionally be non-vanishing on the term corresponding to the matrix

$$\begin{pmatrix}
 & & & & 1 & & & \\
 & & & & \vdots & & & \\
 & & & & n-2 & & & \\
 & & n-1 & & & 3n-4 & \dots & 4n-7 \\
 & & n & & & & & \\
n+1 & \dots & 2n-3 & 3n-6 & 3n-5 & & & \\
 & & 2n-2 & & & & & \\
 & & \vdots & & & & & \\
 & & 3n-7 & & & & &
\end{pmatrix} \tag{3.5}$$

This matrix is derived, associated with the step matrix given by

$$\begin{pmatrix}
 & & & & 1 & 3n-4 & \dots & 4n-7 \\
 & & & & \vdots & & & \\
 & & & & n-2 & & & \\
 & & n-1 & 3n-6 & 3n-5 & & & \\
 & & n & & & & & \\
n+1 & \dots & 2n-3 & & & & & \\
2n-2 & & & & & & & \\
\vdots & & & & & & & \\
3n-7 & & & & & & &
\end{pmatrix}$$

Furthermore, the rows and columns in (3.5) are derived consecutive. Hence, it represents a term in the Saneblidze-Umble diagonal.

However, we can extend our discussion of the combinatorics of the diagonal to recover the forced shape of the matrix for the given inputs for the arities $2(n-2) + m$ and $2(m-2) + n$.

Lemma 3.3.10. *If an element a in a step matrix is a "north-west corner" – i.e. has neighbours down and to the right – then this element cannot be moved under the down and right moves, and it can never be isolated – i.e. alone in both row and column.*

Proof. The element a cannot be moved right, since there is a neighbour in the way. Once the neighbour has moved, we are not allowed to go back and move a afterwards.

Similarily, the neighbour downwards blocks moving a down.

Hence a north-west corner cannot be moved

Suppose now that a were isolated in a derived matrix D associated to a step matrix M in which a was a north-west corner. Suppose further that it had the right neighbour r and the down neighbour d. Then in order for a to be isolated, r must have been moved down and d must have been moved right.

If d moves far enough right not to be in the way for r, then d will move down to the left of r, but then $r > d$ and so the condition for moves that goal rows must have its maximal element smaller than the minimal of the moved subset is broken.

If r moves to the right and then down, then still d will have had to be able to move into the column with r. Thus $d > r$ follows. But then d is blocking the down-move of r.

Hence a north-west corner never becomes isolated. $\square$

All the induced operations of arities $2(n-2)+m$ and $2(m-2)+n$ in $H^*(C_n \times C_m, \mathbb{F}_p)$ will be non-trivial. Indeed, consider the first of these two cases. This arity will contain as a summand the operation given by the snake-like matrix

$$\begin{pmatrix} 1 & & & \\ \vdots & & & \\ (n-2)+1 & n & \dots & (n-2)+(m-2)+1 \\ & & & \vdots \\ & & & 2(n-2)+(m-2)+1 \end{pmatrix}$$

The term described by this matrix won't vanish for the unshuffled arguments given by

$$x|\overset{n}{\dots}|1\overset{m-4}{\dots}|1|x|\overset{n}{\dots}|x \otimes 1|\overset{n-2}{\dots}|1|x|\overset{m}{\dots}|x|1|\overset{n-2}{\dots}|1$$

and this matrix is the only one that will not vanish.

Indeed, the first column is either a run $1, \dots, n-1$, or a singleton from $n, \dots, n+m-4$, or a run $n+m-3, \dots, 2n+m-5$. However, the matrix has exactly one non-singleton row with the same length as the entire matrix width, namely $m-1$. Hence, this row must contain at least the elements $n-1, \dots, n+m-3$. Thus the row is either on the form $x, n-1, \dots, n+m-4$ or $n-1, \dots, n+m-4, x$. If the first column contains a singleton, then the row cannot posisbly be assembled, since then $n-1$ has to come directly to the right of an element $\geq n$. Thus, the first column is a run. It cannot be the latter run for the same reason, thus it will be the run

$1, \ldots, n-1$. However, this run contains the element $n-1$, thus the row has to connect at that point. And, to finish off the argument, the last element of the row will be an element of the column run $n+m-3, \ldots, 2n+m-5$. However, any choice other than setting $x = n+m-3$ will result in a derived non-consecutive matrix, which proves the result.

Chapter 4

Calculation of A_∞ structures

> Formal A_∞-algebras are a bit like abelian groups. If you want to study the internal structure of groups because groups are awesome, then abelian groups seem boring. But when you discover a group in nature that you have no reason to believe is abelian, and it turns out to be, that's telling you something very interesting.
>
> — Ben Webster

4.1 Calculation techniques

Suppose A is a differential graded algebra. Then A is an A_∞-algebra with only μ_1, μ_2 non-trivial. According to the minimality theorem, there is an A_∞-structure for H^*A and an A_∞-quasi-isomorphism $H^* \to A$, and such that in H^*A, μ_1 vanishes and μ_2 is induced by the multiplication in A.

However, the gap between existence of a structure and a reliable description of a specific structure is, as it turns out, rather large. There are several methods established in the literature for calculating an A_∞-structure on H^*A, and even arguments from Johansson and Lambe (2001) that the methods deliver the same structure. We shall here give a somewhat sketchy overview of the main useful calculation methods and their applicability to computing A_∞-structures in group cohomology.

4.1.1 Homological perturbation theory

The exposition here is based to a large part on the discussion in Johansson and Lambe (2001).

The endomorphism dg-algebra $\mathrm{End}_R(pk, pk)$ of a projective resolution pk of the trivial R-module k is quasi-isomorphic to $\mathrm{Ext}^*_R(k, k)$, since homology lifts the projection map $\mathrm{End}_R(pk, pk) \to \mathrm{Ext}^*_R(k, k)$ to an isomorphism.

Now, recall from Section 2.1.4 that an A_∞-structure on $\mathrm{Ext}^*_R(k, k)$ is the same thing as a codifferential on $\bar{T}S\,\mathrm{Ext}^{\geq 1}_R(k, k)$. Hence, what we really do when computing A_∞-structures is to lift this quasi-isomorphism $\mathrm{End}_R(pk, pk) \to \mathrm{Ext}^*_R(k, k)$ to a new quasi-isomorphism $\mathrm{End}_R(pk, pk) \to \bar{T}S\,\mathrm{Ext}^{\geq 1}_R(k, k) \oplus k$ with some additional structure. This extra structure provides us a codifferential on $\bar{T}S\,\mathrm{Ext}^{\geq 1}_R(k, k) \oplus k$, and hence an A_∞-structure on $\mathrm{Ext}^*_R(k, k)$.

With the problem formulated in this way, we can apply the techniques of homological perturbation theory, as developed by Gugenheim and Munkholm (1974); Gugenheim and Stasheff (1986); Gugenheim and Lambe (1989); Gugenheim et al. (1991).

The core idea is that a strong deformation retract (SDR) given by the maps

$$\begin{aligned} f &: \mathrm{End}_R(pk, pk) \to \mathrm{Ext}^*_R(k, k) \\ \nabla &: \mathrm{Ext}^*_R(k, k) \to \mathrm{End}_R(pk, pk) \\ \phi &: \mathrm{End}_R(pk, pk) \to \mathrm{End}_R(pk, pk) \end{aligned}$$

where ϕ is a homotopy between ∇f and the identity on $\mathrm{End}_R(pk, pk)$, might give us enough data to construct an transferred codifferential on $\bar{T}S\,\mathrm{Ext}^*_R(k, k)$. Normally, the maps are required to additionally fulfill

$$\phi^2 = 0 \qquad \phi\nabla = 0 \qquad f\phi = 0$$

In various ways, shown to be equivalent by Johansson and Lambe (2001), these functions are then perturbed to form an SDR with the bar construction on the right hand side, and something reasonably controllable on the left hand side. The formation of the new SDR depends on the *perturbation lemma*. Depending on the construction chosen, different conditions are needed for the collapse of the left hand side back to the dg-algebra $\mathrm{End}_R(pk, pk)$. For details of this construction, see (Gugenheim et al., 1991; Huebschmann and Kadeishvili, 1991).

Similar arguments has been leveraged by Huebschmann (1991) to provide a differential in a spectral sequence converging to $H^*(G, k)$ for metacyclic groups G, and by Berciano (2006) and by Berciano and Umble (2007) to produce explicit descriptions and computational results on the homology $H_*(\pi(\mathbb{Z}, n); \mathbb{F}_p)$.

4.1.2 Merkulov and splitting the chain algebra

Sergei Merkulov worked out an explicit expression for the A_∞-structure maps and the corresponding quasi-isomorphism while discussing the use of A_∞-techniques in the study of Kähler manifolds, see (Merkulov, 1999). The exposition here takes some choices from the exposition by Lu et al. (2006).

The setup requires a vector space splitting $A = H \oplus B \oplus L$, where $H \cong H^*A$, $B \cong \operatorname{Im}\partial$ and L is the corresponding k-linear complement. We identify $H^*A = H$ over the isomorphism between these. Let π_H be the canonical projection $A \to H$ and let $G : A \to A$ be a homotopy from Id_A to π_H. Thus $\mathrm{Id}_A - \pi_H = \partial G + G\partial$. We further wish to choose the homotopy G with some care – namely such that $G_n : A_n \to A_{n-1}$ satisfies $G_n|_{L_n} = 0$, $G_n|_{H_n} = 0$ and $G_n|_{B_n} = (\partial_{n-1}|_{L_{n-1}})^{-1}$.

Then, we have $G_n A_n = L_{n-1}$, $G_{n+1}\partial_n = \pi_{L_n}$ and $\partial_{n-1}G_n = \pi_{B_n}$. We can define $G\lambda_1 = -\mathrm{Id}_A$, pick λ_2 to be multiplication on A and define λ_n for $n \geq 3$ by

$$\lambda_n = \sum_{s+t=n} (-1)^{s+t} \lambda_2(G\lambda_s \otimes G\lambda_t)$$

Then the family of maps $m_i = \pi_H \lambda_i$ form an A_∞-algebra structure on H^*A according to the minimality theorem, and the family of maps $f_i = -G\lambda_i$ give us the expected quasi-isomorphism of A_∞-algebras.

We can choose the A_∞-algebra structure to be strictly unital, as long as we pick H^0 to contain the unit of A.

In essence, this construction gives the SDR data needed for the homology perturbation theory computations so explicitly that the resulting formulae can be written down explicitly.

4.1.3 The Kadeishvili algorithm

Consider the proof of the minimality theorem given in Section 2.2.1. The proof suggests an algorithm for computing an A_∞-structure on H^*A together with a quasi-isomorphism $H^*A \to A$:

For the computation of an A_∞-structure on H^*A, we need to fix some data for the entire computation. Central to this is the choice of a cycle-choosing map $f_1 \colon H^*A \to A$. We need the map to send classes to cycles representing the classes, but any such choice will work.

The algorithm takes as input a list of elements $a_1, \dots, a_n$ in H^*A, a cycle-choosing map $f_1 \colon H^*A \to A$ as described above, and returns $m_n(a_1, \dots, a_n)$ and $f_n(a_1, \dots, a_n)$ fulfilling the Stasheff axioms St_n and St_n^m.

1. If $n = 1$, return $m_1(a_1) = 0$ and $f_1(a_1)$ immediately.

2. If $n = 2$, set $\Psi_2(a_1, a_2) = f_1(a_1)f_1(a_2)$ and $m_2(a_1, a_2) = a_1a_2$ and go to step 4. Otherwise, compute

$$\begin{aligned}\Psi_n(a_1, \dots, a_n) = \\ &\sum_{s=1}^{n-1} (-1)^{\varepsilon_1(a_1,\dots,a_n,s)} f_s(a_1, \dots, a_s) \cdot f_{n-s}(a_{s+1}, \dots, a_n) + \\ \sum_{j=2}^{n-1} \sum_{k=0}^{n-j} &(-1)^{\varepsilon_2(a_1,\dots,a_n,k,j)} f_{n-j+1}(a_1, \dots, a_k, m_j(a_{k+1}, \dots, a_{k+j}), \dots, a_n)\end{aligned}$$

   where the expressions $\varepsilon_1(a_1, \dots, a_n, s) = s + (n - s + 1)(|a_1| + \cdots + |a_s|)$ and $\varepsilon_2(a_1, \dots, a_n, k, j) = k + j(n - k - j + |a_1| + \cdots + |a_k|)$ are the signs in the Stasheff morphism axiom St_n^m with the Koszul signs introduced.

   Note that the values of f_k and m_k for $k < n$ may be computed recursively using subsequent calls to this algorithm. The recursion bottoms out since m_1, m_2 and f_1 are already given.

3. By the proof of the minimality theorem, the element $\Psi_n(a_1, \dots, a_n) \in A$ is a cycle. Hence, it belongs to some homology class x. Set $m_n(a_1, \dots, a_n) = x$.

4. Since $m_n(a_1, \dots, a_n)$ is the homology class containing $\Psi_n(a_1, \dots, a_n)$, the representing cycle $f_1(m_n(a_1, \dots, a_n))$ is homologous to the cycle $\Psi_n(a_1, \dots, a_n)$. Thus $\Psi_n(a_1, \dots, a_n) - f_1(m_n(a_1, \dots, a_n))$ is a boundary, and we can pick an element y such that $dy = \Psi_n(a_1, \dots, a_n) - f_1(m_n(a_1, \dots, a_n))$. We set $f_n(a_1, \dots, a_n) = y$, and we return the higher multiplication $m_n(a_1, \dots, a_n)$ and the quasi-isomorphism component $f_n(a_1, \dots, a_n)$

For the specific case of computing $\mathrm{Ext}_R^*(k, k)$ as the homology of the dg-algebra $\mathrm{End}_R(pk, pk)$, we note that homologous maps in $\mathrm{End}_R(pk, pk)$ are chain homotopic maps, that cycles are chain maps and that boundaries are null-homotopic, with the preimages being the null-homotopies.

Hence, the algorithm works by picking null-homotopies for graded endomorphisms of a resolution.

If we can find R-algebra generators for $\mathrm{Ext}^*_R(k,k)$, it makes the computation easier to choose a basis for $\mathrm{Ext}^*_R(k,k)$ consisting of monomials in a set of generators, and equipped with a preferred factorisation of the basis elements. We can then define f_1 on these generators and extend multiplicatively to all basis elements and linearly to all of $\mathrm{Ext}^*_R(k,k)$. In the example of the cyclic groups, see Section 4.3.3, we make essential use of a system of parameters of length 1, such that $\mathrm{Ext}^*_R(k,k)$ is free of rank 2 over the polynomial algebra of the parameters.

4.2 Global vs. local computation

As long as the dg-algebras we study are easily described, or we can use auxiliary methods to find the decompositions necessary for Merkulov's technique or for using HPT, the issue of computing an A_∞-structures is mainly an issue of applying known techniques and assembling the information required as input.

However, for the case of group cohomology, the easiest dg-algebra to use is $\mathrm{End}_{kG}(pk, pk)$ for a minimal projective resolution pk of the trivial module k. Even for small groups, it is not obvious that the algebra need even be degreewise finite dimensional, and any sufficiently adequate description of the algebra that allows the use of the more powerful computational methods available seems far from reachable.

4.3 Black-box computation

Given the limitations sketched in the previous section, I would propose using the Kadeishvili algorithm, if not to gain a complete description, then at least in order to gain intuition and insight into the structures found. Both the homology perturbation theory approach and Merkulov's approach above require global information about the algebraic structure under consideration, and computations done using global decompositions. By instead performing computations internally to the dg-algebra A, we can view it as a computational black box, delivering answers when asked, but not sharing any information spontaneously. By using the Kadeishvili algorithm we can extract single higher multiplication maps this way. Extracting a full A_∞-structure, however, is much more cumbersome, and relies, in the cases where I can do it, on computational reduction results.

I have written an implementation of this approach as a module for the computer algebra system MAGMA (Bosma, Cannon, and Playoust, 1997). The implementation computes, inductively, a strictly unital A_∞-algebra structure on $H^*(G, \mathbb{F}_p)$. See Example 4.3.1 for a computation example.

The module centers on three main functions:

`AInfinityRecord`, which takes a p-group G and an integer n, constructs a free resolution of k with kG-modules of length n, computes a partial ring structure on $H^{\leq n}(G,k)$ and stores all of this in the return value from `AInfinityRecord`. Thus, if `Aoo` is such a return value, then the following members might be of interest for a user:

`Aoo‘P` The projective resolution

`Aoo‘R` The polynomial ring generated by a minimal set of generators for $H^{\leq n}(G,k)$.

`Aoo‘S` The ring induced by the generators and relations detected by $H^{\leq n}(G,k)$ as a quotient ring structure.

`HighProduct` takes the A_∞-record `Aoo` constructed by a call to the function `AInfinityRecord` and a list of elements $s_1, \ldots, s_n$ from `Aoo‘S`, and returns the higher product $m_n(s_1, \ldots, s_n)$ in the induced A_∞-structure on $H^*(G,k)$. An error message is emitted if the resolution length is insufficient for the computation at hand. The computation as such works lazily, caching intermediate results. Since the algorithm has such high degree of recursion, this caching speeds up some of the computations.

`HighMap` takes the A_∞-record `Aoo` constructed by `AInfinityRecord` and a list of elements $s_1, \ldots, s_n$ from `Aoo‘S`, and returns the component $f_n(s_1, \ldots, s_n)$ of the calculated quasi-isomorphism $H^*(G,k) \to \mathrm{End}(P)$. Internally, `HighMap` and `HighProduct` call the same common computational code, and rely on the caching to keep the results around to the function call.

4.3.1 Computational reduction

At a first glance, computing A_∞-structures by a blackbox method seems infeasible due to the high degree of recursion of the computations and the multiple infinities involved: there are infinitely many arities to compute, and even H^*A tends to be infinite dimensional in group cohomology, hence so is $(H^*A)^{\otimes n}$ for all the relevant values of n.

Example 4.3.1: Usage example for the MAGMA module computing an A_∞-operation on $H^*(C_3, \mathbb{F}_3)$.

```
$ magma
[0]
Magma V2.14-9      Fri Feb 15 2008 11:53:02 on mpc721
[Seed = 3576167808]
Type ? for help.  Type <Ctrl>-D to quit.

Loading startup file "/home/mik/.magmarc"

> Aoo := AInfinityRecord(CyclicGroup(3),6);
> S<x,y> := Aoo`S;
> HighProduct(Aoo,[x,x,x]);
y
> ModuleMaps(HighMap(Aoo,[x,x]));
[*
    [2 0 0]
    [0 2 0]
    [0 0 2],

    [0 0 0]
    [0 0 0]
    [0 0 0],

    [2 0 0]
    [0 2 0]
    [0 0 2],

    [0 0 0]
    [0 0 0]
    [0 0 0],

    [2 0 0]
    [0 2 0]
    [0 0 2],

    [0 0 0]
    [0 0 0]
    [0 0 0]
*]
```

In specific cases, however, we are able to reduce the complexity of these computations to more easily handled sizes. For especially well-behaved cohomology rings we are able to reduce the computation of a full A_∞-structure to a finite problem.

Lemma 4.3.2. *Suppose that R is a finite k-algebra and that*

A1. $A = \mathrm{End}_R(X)$ is the endomorphism dg-algebra of some complex of finitely generated R-modules. Suppose further that there is an element $z \in H^*A$ generating a polynomial subalgebra and that H^*A is free as a $k[z]$-module.

A2. we have chosen m_k and f_k for all $k < n$ such that for all $a_1, \dots, a_k \in H^*A$, and $\zeta = f_1(z)$,

- *Each $f_1(a)$ is a cocycle representing a*
- $f_1(za) = \zeta f_1(a)$
- $m_k(a_1, \dots, za_i, \dots, a_k) = zm_k(a_1, \dots, a_k)$
- $f_k(a_1, \dots, za_i, \dots, a_k) = \zeta f_k(a_1, \dots, a_k)$
- $\zeta f_k(a_1, \dots, a_k) = f_k(a_1, \dots, a_k)\zeta$

A3. $b_1, \dots$ is a $k[z]$-basis of H^*A, and that we have chosen $m_n(v_1, \dots, v_n)$ and $f_n(v_1, \dots, v_n)$ according to the Kadeishvili algorithm for all $v_i \in \{b_1, \dots\}$.

Then we can choose m_n and f_n for values in all of H^*A according to the Kadeishvili algorithm such that $m_n(a_1, \dots, za_i, \dots, a_n) = zm_n(a_1, \dots, a_n)$ and such that $f_n(a_1, \dots, za_i, \dots, a_n) = \zeta f_n(a_1, \dots, a_n)$.

Proof. We need to consider

$$\begin{aligned}\Psi_n(a_1, \dots, za_i, \dots, a_n) = \sum \pm f_j(a_1, \dots, za_i, \dots, a_j) f_{n-j}(a_{j+1}, \dots, a_n) + \\ \sum \pm f_j(a_1, \dots, a_j) f_{n-j}(a_{j+1}, \dots, za_i, \dots, a_n) + \\ \sum \pm f_{n-j-1}(a_1, \dots, za_i, \dots, m_j(a_{k+1}, \dots, a_{k+j}), \dots, a_n) + \\ \sum \pm f_{n-j-1}(a_1, \dots, m_j(a_{k+1}, \dots, za_i, \dots, a_{k+j}), \dots, a_n) + \\ \sum \pm f_{n-j-1}(a_1, \dots, m_j(a_{k+1}, \dots, a_{k+j}), \dots, za_i, \dots, a_n) \quad .\end{aligned}$$

In each summand of this expression, the term za_i occurs within either a f_j or a m_j of lower arity than n. Hence, by assumption, we can commute z out to a ζ. Since, also, ζ commutes with all f_n of lower arity, we find that

$$\Psi_n(a_1, \dots, za_i, \dots, a_n) = \zeta \Psi_n(a_1, \dots, a_n) \quad .$$

Hence $m_n(a_1, \dots, za_i, \dots, a_n) = zm_n(a_1, \dots, a_n)$ follows. We need, to finalize the argument, to find a null-homotopy h of the map $\zeta(\Psi_n - f_1 m_n)(a_1, \dots, a_n)$ given a null-homotopy h' of $(\Psi_n - f_1 m_n)(a_1, \dots, a_n)$. As

$$\zeta(\Psi_n - f_1 m_n)(a_1, \dots, a_n) = \zeta(dh' + h'd) = d(\zeta h') + (\zeta h')d$$

where $\zeta d = d\zeta$ holds because $\zeta = f_1(z)$ and all images of elements in H^*A in A under f_1 can be chosen to be chain maps representing their equivalence classes in H^*A, such a null-homotopy is given by $h = \zeta h'$. □

Lemma 4.3.3. *Suppose that R is a finite k-algebra and that*

B1. X is a periodic resolution of period π of finitely generated R-modules, and that $A = \mathrm{End}_R(X)$ is the endomorphism dg-algebra of X. Suppose further that there is some element $0 \neq z \in H^*A$ such that we can choose $f_1(z) = \zeta$, a periodic map of period π with each $\zeta_n = \mathrm{Id}$.

B2. for all $k < n$ we have constructed m_k and f_k such that A2 holds.

B3. $b_1, \dots, b_t$ is a $k[z]$-basis for H^*A and for all $v_1, \dots, v_n$ chosen such that each $v_i \in \{b_1, \dots, b_t\}$ we know that $f_n(v_1, \dots, v_n)$ is periodic of period π for all choices $v_1, \dots, v_n$.

Then, from B1, we can infer that z generates a polynomial subalgebra of H^*A and H^*A is free over $k[z]$.

From B1 and B2 we can conclude, using $\zeta f_k(a_1, \dots, a_k) = f_k(a_1, \dots, a_k)\zeta$, that all the homotopies $f_n(a_1, \dots, a_k)$ are periodic of period π.

From B3 we can additionally conclude for all $a_1, \dots, a_n \in H^*A$, that the map $f_n(a_1, \dots, a_n)$ is periodic of period π and

$$\begin{aligned} m_n(a_1, \dots, za_i, \dots, a_n) &= zm_n(a_1, \dots, a_n) \\ f_n(a_1, \dots, za_i, \dots, a_n) &= \zeta f_n(a_1, \dots, a_n) \\ f_n(a_1, \dots, a_n)\zeta &= \zeta f_n(a_1, \dots, a_n) \quad . \end{aligned}$$

Proof. If some ζ^N would be null-homotopic, we can use the periodicity of X to shift the null-homotopy down in degree. Hence, such a null-homotopy induces a null-homotopy for ζ. However, we assumed that $z \neq 0$. Hence $f_1(z) = \zeta$ is not null-homotopic. Thus, $k[z]$ is a polynomial subalgebra of H^*A.

We set $I = H^{\geq 1}A$. This is an ideal in H^*A, and we can find the k-vector space $J = I/I^2$ of indecomposables. We can pick a basis $b_1, \dots, b_r$ of $J/(z)$. Every b_i has a representative in J, hence a representative that is not divisible by z. Furthermore, $b_1, \dots, b_r, z$ generate H^*A as a k-algebra.

Suppose now that we had some dependency $\sum_i a_i b_i = 0$ over $k[z]$. Then $z|a_i$ for all a_i, since otherwise the b_i would not form a basis of $J/(z)$. But then we could divide the dependency by an appropriate power of z and get a dependency involving the indecomposables. Hence H^*A is free over $k[z]$.

From the condition $f_k(a_1,\dots,a_k)\zeta = \zeta f_k(a_1,\dots,a_k)$ we get by setting $d = |f_k(a_1,\dots,a_k)|$, that $(f_k(a_1,\dots,a_k)\zeta)_n = f_k(a_1,\dots,a_k)_n\zeta_{n+d}$ and that $(\zeta f_k(a_1,\dots,a_k))_n = \zeta_n f_k(a_1,\dots,a_k)_{n+\pi}$. Equality of chain maps forces the equality $f_k(a_1,\dots,a_k)_n\zeta_{n+d} = \zeta_n f_k(a_1,\dots,a_k)_{n+\pi}$, and by the definition of ζ, we are left with $f_k(a_1,\dots,a_k)_n = f_k(a_1,\dots,a_k)_{n+\pi}$.

Since $b_1,\dots,b_t$ form a $k[z]$-linear basis of H^*A, any element $a \in H^*A$ has a unique decomposition into a $k[z]$-linear combination of the b_i.

By Lemma 4.3.2, all the commutativity relations hold.

For periodicity of $f_n(a_1,\dots,a_n)$, consider the terms of the difference $\Psi_n(a_1,\dots,a_n) - f_1 m_n(a_1,\dots,a_n)$. Each term in this expression is either a composition of periodic maps of period π, or a periodic map of period π, by the assumptions on all f_k. Hence, $\Psi_n(a_1,\dots,a_n) - f_1 m_n(a_1,\dots,a_n)$ is periodic of period π.

Finally, by assumption B3, $f_n(v_1,\dots,v_n)$ is periodic of period π, for all choices of $v_1,\dots,v_n \in \{b_1,\dots,b_t\}$. Hence $f_n(v_1,\dots,zv_k,\dots,v_n) = \zeta f_n(v_1,\dots,v_n)$, and by Lemma 4.3.2, the resulting homotopy $f_n(a_1,\dots,a_n)$ is given by composing some ζ^s, which has period π, with $f_n(v_1,\dots,v_n)$, which is also periodic of period π. □

Lemma 4.3.4. *Let A be a dg-algebra. Suppose that in the computation of an A_∞-structure on H^*A, we have been able to show that $f_k = 0$ and $m_k = 0$ for all $q \le k \le 2q-2$ for some q.*

Then $f_k = 0$ and $m_k = 0$ for all $k \ge q$.

Proof. The proof follows by induction. Suppose that $\kappa > 2q-2$, and that we have already proven $f_k = 0$ and $m_k = 0$ for all $q \le k < \kappa$. In the computational step where we compute f_κ and m_κ, we start by considering Ψ_κ. This expression has two kinds of terms.

First, there are the terms of the form $f_i \cdot f_{\kappa-i}$. Since $\kappa > 2q-2$, either $i \ge q$ or $\kappa - i \ge q$. Hence by the induction hypothesis, $f_i \cdot f_{\kappa-i} = 0$.

Second, there are the terms of the form $f_i \circ_j m_{\kappa-i+1}$. Again, either $\kappa - i + 1 \ge \kappa - i \ge q$ or $i \ge q$. Hence, by hypothesis either $f_i = 0$ or $m_{\kappa-i+1} = 0$.

Hence $\Psi_\kappa = 0$. Thus we can choose $m_\kappa = 0$ and $f_\kappa = 0$. This shows the induction step and concludes the proof. □

4.3.2 Minimal complexity cohomology rings

For a discussion of complexity of cohomology rings as such, we refer to Carlson et al. (2003, Chapter 10). We shall recall a few core concepts here.

Definition 4.3.5. Write $H(G) = H^*(G,k)$ if k has characteristic 2 and $H(G) = H^{\text{even}}(G,k)$ otherwise.

We note that $H(G)$ is a subring of $H^*(G,k)$.

Definition 4.3.6. The *complexity* of a cohomology ring $E(M) = \operatorname{Ext}^*_R(M,M)$ is the least integer s such that

$$\lim_{n\to\infty} \frac{\dim_k E(M)^n}{n^s} = 0 \quad .$$

We denote the complexity of $E(M)$ by $c(E(M))$.

Note also that $c(E(M)) = \dim_{\text{Krull}} H^*(G,k)/J(M)$ where we define $J(M) = \operatorname{Ann}_{H(G)} E(M)$. Specifically we know from this that $c(E(k)) = \dim_{\text{Krull}} H^*(G,k)$.

Theorem 4.3.7. *Suppose $H^*(G,k)$ has complexity 1. Then there is some integer n such that we can pick a $\gamma \in H^n(G,k)$ that is represented by a degree shifted identity map on the minimal resolution P_* of k.*

Proof. Proposition 8.4.4 in (Evens, 1991) proves that k has a periodic minimal projective resolution P_*. Benson (2001) provides in his Theorem 2.1 as a known, but apparently not previously stated, fact that the depth of $H^*(G,k)$ is at least the depth of $H^*(P,k)$ for P a Sylow p-subgroup of G. Duflot (1981) proves that the depth of $H^*(P,k)$ is at least the p-rank of $Z(P)$, which for a p-group is at least 1.

Hence, the depth of $H^*(G,k)$ is at least 1 due to Duflot and Benson, and it is at most 1, since the depth of a ring is at most the Krull-dimension of the ring. Hence $H^*(G,k)$ is Cohen-Macaulay and contains a regular element γ of degree j, say. By choosing $\chi = \gamma$ in Evens' proof of his Proposition 8.4.4 one sees that j is a period of P_* and that $H^*(G,k)$ is a finite $k[\gamma]$-module. Duflot (1981) tells us that this γ will be injective as a map $H^n(G,k) \to H^{n+j}(G,k)$. Since this is an injection of finite dimensional k-vector spaces of identical dimension, it has to be bijective, hence an isomorphism. □

The result can be proven by a minor modification of Proposition 8.4.4 in (Evens, 1991), however we chose for economy of exposition to use the more powerful results from Benson (2001) and Duflot (1981).

Using Theorem 4.3.7, we see that for groups of cohomological complexity 1, $H^*(G,k)$ fulfills the requirements B1 and B2 of Lemma 4.3.3. Thus, we are in a position where we can hope to reduce the computational load for finding an A_∞-structure on $H^*(G,k)$ to a finite computation. While doing this, we need to take care to compute all higher operations and quasi-isomorphism component maps of the $k[z]$-basis elements of $H^*(G,k)$ for each single arity in turn. If we can show that the quasi-isomorphism component map images have unbounded periodicity, then the requirement B3 of Lemma 4.3.3 fails.

4.3.3 The cohomology of a cyclic group

Consider $G = C_q$, with $q = p^r$ for some prime p and some r. For simplicity, we choose $q > 2$. Over $\mathbb{F}_pG$, the simple $\mathbb{F}_p$ has a particularly nice minimal resolution. With $G = \langle g \rangle$, we set $\alpha = g - 1$. Then kG has the finite presentation $k[\alpha]/(\alpha^q)$. Note that $\alpha \cdot \alpha^{q-1} = \alpha^{q-1} \cdot \alpha = 0$ in $\mathbb{F}_pG$. The minimal resolution then has the guise

$$\ldots \longrightarrow kG \xrightarrow{\cdot\alpha^{q-1}} kG \xrightarrow{\cdot\alpha} kG \xrightarrow{\cdot\alpha^{q-1}} kG \xrightarrow{\cdot\alpha} kG \xrightarrow{\epsilon} k \longrightarrow 0$$

Thus, $H^n(G,\mathbb{F}_p) \cong \mathbb{F}_p$ for all $n \geq 0$. Now, we can start figuring out the product structure. First off, we consider what the chain map representatives of the cohomology classes would look like. Since the dimension $\dim H^1(G,\mathbb{F}_p) = 1$, we know that there is one generator x of degree 1. One way to compute a chain map representative of this coclass is to take the know non-trivial map $\epsilon\colon kG \to k$, and lift it to a chain map ξ. We get

$$\begin{tikzcd}
\ldots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] \arrow[ddr, "\epsilon"] & kG \\
\xi : & & & & & \\
\ldots \arrow[r] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k
\end{tikzcd} .$$

If we try to square this generator, we would compute

$$\begin{tikzcd}
\ldots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r] \arrow[dd, "\cdot 1"] \arrow[ddr, "\epsilon"] & kG \\
\xi : & & & & & \\
\ldots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] \arrow[ddr, "\epsilon"] & kG \arrow[r, "\epsilon"] & k \\
\xi : & & & & & \\
\ldots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}$$

ending up with the degree 2 chain map consisting of multiplication with α^{q-2} in every component. Now, the composition $\epsilon \circ (\cdot\alpha^{q-2})$ vanishes, since $\epsilon\alpha = \epsilon(g-1) = 1 - 1 = 0$. Thus x squares to 0 in the cohomology ring.

From this, we can conclude two useful things. First off, $x^2 = 0$ will be a relation in the presentation of $H^*(G, \mathbb{F}_p)$ as a ring. Furthermore, there is a new generator y of degree 2. We can compute a chain map representative η of y by extending $\epsilon\colon kG \to k$ to a chain map as follows

$$\begin{tikzcd}
 & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] \arrow[dr, "\epsilon"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r] & k \arrow[r] & 0 \\
\eta : & & & & & & & & & \\
 & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0 & &
\end{tikzcd} .$$

Thus, composition with η is just a dimension shift of the complex, and so, with compositions of x and y, we get one non-trivial coclass in every dimension. From this, we can conclude that $H^*(G, \mathbb{F}_p) = k[x, y]/(x^2)$ as a graded commutative ring.

As for the induced A_∞-structure on $H^*(G, \mathbb{F}_p)$, we shall compute it following a computation by Madsen (2002).

Note that the resolution given above will work, as written, for the more generic case of a graded algebra $k[\gamma]/(\gamma^q)$ for γ in degree m, since the degree of γ does not really enter into the calculation of the resolution. We get two different gradings – each module being graded in the monoid $m\mathbb{Z}/mq\mathbb{Z}$, and the resolution being graded by homological degrees.

In order to compute an A_∞-structure on the cohomology ring, we start out by setting $m_1 = 0$ and m_2 to the product in $H^*(G, \mathbb{F}_p)$. We further fix f_1 to pick out the representatives displayed above for the coclasses, with $f_1(y^n)$ being the chain map that is identity on each component and drops homological degree by $2n$, and $f_1(xy^n)$ the composition $f_1(y^n) \circ \xi$.

This structure turns out to be nice enough for the computation of the A_∞-structure to be significantly simplified. The cohomology ring has complexity 1 and the element y generates a polynomial subalgebra and fulfills the condition B1 in Lemma 4.3.3. Hence we can start computing the A_∞-structure maps in increasing arity, working with only the $k[y]$-basis of $H^*(G, \mathbb{F}_p)$ – in other words only with 1 and x – as argument to the operations. As long as the thus computed f_k turn out to be periodic of period dividing 2 (the period of η), we can keep on computing and be certain that the computation reveals the structure we're searching for.

Should 1 occur as argument somewhere for a higher multiplication – with $n > 2$ – then we can choose both f_n and m_n equal to zero for that argument since the minimality theorem includes strict unitality. Thus, it remains to compute $f_n(x, x, \dots, x)$ and $m_n(x, x, \dots, x)$. We can, due to Lemma 4.3.3, do so as long as we never, while climbing up the arities, encounter a null-homotopy that is not periodic of period dividing 2.

Considering

$$\Psi_2(x, x) = f_1(x) f_1(x) - f_1(x^2) = -f_1(x)^2$$

as a chain map we find

$$\begin{tikzcd}
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\xi^2 : \cdot(-\alpha^{q-2})"'] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot(-\alpha^{q-2})"'] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-\alpha^{q-2})"'] \arrow[dr, "\epsilon"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r] & k \arrow[r] & 0 \\
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd} .$$

So we need, for $f_2(x, x)$, to find a null-homotopy for the chain map with $m \mapsto (-\alpha^{q-2}) \cdot m$ in each degree. One such null-homotopy is given by the map

$$\begin{tikzcd}
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-\alpha^{q-3})"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-\alpha^{q-3})"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & kG \\
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k
\end{tikzcd} \quad h : .$$

Indeed, $dh + hd$ is the map which multiplies an odd-degree component of the resolution by $0\alpha^{q-1} + (-\alpha^{q-3})\alpha = -\alpha^{q-2}$. An even-degree component gets multiplied by $\alpha(-\alpha^{q-3}) + 0\alpha^{q-1} = -\alpha^{q-2}$.

We note that for each k, the map $\Psi_k(x, x, \dots, x)$ will have the same degree as $m_k(x, x, \dots, x)$. This degree is $2 - k + k \cdot |x| = 2$, and hence the homotopy h will always have degree 1, and so the homotopy differential is $dh - (-1) \cdot hd = dh + hd$.

Thus, we can continue, with each $f_i(x, x, \dots, x)$ consisting of a map alternating between the 0 map and $-\alpha^{q-1-i}$. Each of these maps belong to the homology class of $0 \in \mathrm{Ext}^*_R(k, k)$. Thus, each $m_i(x, x, \dots, x) = 0$.

At this point, it is time to consider how this process finishes. For the quasi-isomorphism component $f_{q-1}(x, x, \dots, x)$ we get the chain map

$$\begin{tikzcd}
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-1)"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-1)"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & kG \\
\dots \arrow[r] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k
\end{tikzcd} \quad h : .$$

Thus, when computing $\Psi_q(x, x, \ldots, x)$, we get the result

$$\begin{aligned}\Psi_q(x, x, \ldots, x) = \sum \pm f_i(x, x, \ldots, x) f_{q-i}(x, x, \ldots, x) + \\ + \sum \pm f_i(x, x, \ldots, m_{q-i-1}(x, x, \ldots, x), \ldots, x)\end{aligned}$$

whereby all the m_i vanish. This leaves us with only the products of f_i. For all $i < q-1$, we get the composition

$$\begin{tikzcd}
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-\alpha^s)"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-\alpha^s)"] & kG \arrow[r] \arrow[dd, "\cdot 0"] \arrow[ddr, "\epsilon"] & kG \\
f_{q-i}(x, \ldots, x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-\alpha^t)"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] \arrow[ddr, "\epsilon"] & kG \arrow[r, "\epsilon"] & k \\
f_i(x, \ldots, x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}$$

for some s, t, and hence the composition vanishes. We get the remaining summands $\Psi_q(x, \ldots, x) = -f_1(x) f_{q-1}(x, \ldots, x) - f_{q-1}(x, \ldots, x) f_1(x)$. These work out to the compositions

$$\begin{tikzcd}
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-1)"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-1)"] & kG \arrow[r] \arrow[dd, "\cdot 0"] \arrow[ddr, "\epsilon"] & kG \\
f_{q-1}(x, \ldots, x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] \arrow[ddr, "\epsilon"] & kG \arrow[r, "\epsilon"] & k \\
f_1(x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot\alpha^{q-2}"] & kG \arrow[r] \arrow[dd, "\cdot 1"] \arrow[ddr, "\epsilon"] & kG \\
f_1(x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[dd, "\cdot(-1)"] & kG \arrow[r, "\cdot\alpha"] \arrow[dd, "\cdot 0"] \arrow[ddr, "\epsilon"] & kG \arrow[r, "\epsilon"] & k \\
f_{q-1}(x, \ldots, x) : & & & & & \\
\cdots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}$$

.

Taking together the compositions in both diagrams, we arrive at the chain map

$$
\begin{tikzcd}
{} \arrow[d, phantom, "\Psi_q(x,\dots,x):"] & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] \arrow[dr, "\epsilon"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r] & k \arrow[r] & 0 \\
{} & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}
.
$$

We recognize this as not only a chain map belonging to the coclass y, but the particular chain map $f_1(y)$. Thus, we can choose the zero map as our null-homotopy $f_q(x,\dots,x)$.

The images of the m_k computed thus far are all either 0 or some power of y. Hence, by the computation thus far, the extra condition on the A_∞-structure in Lemma 4.3.4 holds, and we can detect the vanishing of all f_k and m_k for $k > q$ by computing f_k and m_k for $q+1 \le k \le 2q$.

As an induction step, we consider some $\Psi_{q+j}(x,\dots,x)$ and assume additionally that $f_k = 0$ and $m_k = 0$ for $q < k < q+j$. This expression has terms of two types: of the type $f_i(x,\dots,x)f_{q+j-i}(x,\dots,x)$ and of the type $f_i(x,\dots,x,m_{q+j-1-i}(x,\dots,x),x,\dots,x)$.

If $i \ge q$ or $j-i \ge 0$, then the corresponding term of the first type vanishes. The remaining terms of the first type will have $i < q$ and $j < i$. These terms are all on the form

$$
\begin{tikzcd}
{} \arrow[d, phantom, "f_{q+j-i}(x,\dots,x):"] & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot\alpha^s"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot\alpha^s"] & kG \arrow[r] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & kG \\
{} \arrow[d, phantom, "f_i(x,\dots,x):"] & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & kG \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot\alpha^t"] & kG \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & kG \arrow[r, "\epsilon"] & k \\
{} & \dots \arrow[r] & kG \arrow[r, "\cdot\alpha^{q-1}"] & kG \arrow[r, "\cdot\alpha"] & kG \arrow[r, "\epsilon"] & k \arrow[r] & 0
\end{tikzcd}
$$

for appropriate t, s, and it's easy to convince ourselves that all these terms vanish.

The terms of the second type all vanish since $m_{q+j-1-i}(x,\dots,x)$ is either 0 or some power of y. Hence the composition vanishes.

Thus, $\Psi_{q+j}(x,\dots,x) = 0$, and we can choose $m_{q+j}(x,\dots,x) = 0$ and $f_{q+j}(x,\dots,x) = 0$. This holds for all $0 \le j \le q+2$, and thus by Lemma 4.3.4, all operations and quasi-isomorphism components of arity more than q vanish.

This demonstrates the following theorem:

Theorem 4.3.8 (Madsen (2002)). *Suppose $p|n$ and $n \geq 3$. Then $H^*(C_n, \mathbb{F}_p)$ has an A_∞-structure given by m_2 being the usual cup product and the n-ary $m_n(xy^{e_1}, \ldots, xy^{e_n}) = y^{1+\sum e_i}$ being the only non-vanishing operations.*

4.3.4 Partial computations

Even if we don't have the situation of Section 4.3.2, where we can justify brute force searches, we can still produce hints of the structure by specific computation.

I will in this exposé consider three different 2-groups, and work out hints of an A_∞-structure on their cohomologies. The groups are D_8, the dihedral group on eight elements, D_{16}, the dihedral group on 16 elements and Q_8, the quaternionic unit group.

The cohomology of D_8

The cohomology ring of D_8 has a presentation given by $H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy)$, with $|x| = |y| = 1$ and $|z| = 2$. The group algebra is given by the presentation $\mathbb{F}_2[a, b]/(a^2, b^2, abab + baba)$. From this, we expect, using the higher multiplication theorem, the operations $m_2(x, x)$, $m_2(y, y)$, $m_4(x, y, x, y)$ and $m_4(y, x, y, x)$ to be non-trivial, of degree 2 and such that $m_4(x, y, x, y) = m_4(y, x, y, x)$. The computation in Example 4.3.9 confirms these expectations.

Example 4.3.9: Higher multiplications implied from the higher multiplication theorem on $H^*(D_8, \mathbb{F}_2)$

```
> G := DihedralGroup(4);
> Aoo := AInfinityRecord(G,10);
> S<x,y,z> := Aoo'S;
> HighProduct(Aoo,[x,x]);
x^2
> HighProduct(Aoo,[y,y]);
y^2
> HighProduct(Aoo,[x,y,x,y]);
z
> HighProduct(Aoo,[y,x,y,x]);
z
```

By experimentation, we can verify further interesting properties of these operations. First off, we can conjecture that there are infinitely many elements of $H^*(D_8, \mathbb{F}_2)^{\otimes 4}$ such that m_4 doesn't vanish.

Computation 4.3.10. We find that for $1 \leq n \leq 10$,

$$m_4(x^n, y, x, y) = m_4(y, x, y, x^n) = x^{n-1}z$$
$$m_4(y^n, x, y, x) = m_4(x, y, x, y^n) = y^{n-1}z$$

This inspires the following conjecture:

Conjecture 4.3.11. *There is an A_∞-structure on $H^*(D_8, \mathbb{F}_2)$, that can be computed using an implementation of the Kadeishvili algorithm such that for all $n \geq 1$:*

$$m_4(x^n, y, x, y) = m_4(y, x, y, x^n) = x^{n-1}z$$
$$m_4(y^n, x, y, x) = m_4(x, y, x, y^n) = y^{n-1}z$$

Computation 4.3.12. It does not, however, seem as if z works similar enough to the behaviour of the special coclass required in Lemma 4.3.2. One example indicating this is given by computing $m_3(y, x, x) = 0$ and $m_3(y, x, xz) = x^2z$.

Computation 4.3.13. We can verify, using brute force computation, that $H^*(D_8, \mathbb{F}_2)$ has non-trivial higher multiplications and non-trivial higher components of the companion quasi-isomorphism in what seems to be all higher arities. Specifically, by computing all m_i and f_i applied to degree 1 arguments, for $i \leq 7$, we find non-trivial values in all arities $2 \leq i \leq 7$.

Studying random samples of the computed operations, and experimenting slightly with argument patterns, we ended up computing the maps $m_n(x, y, y, \dots, y, x)$ for various n. $m_1(x) = 0$ and $m_2(x, x) = x^2$ are inherent in the way we construct the A_∞-structure. $m_3(x, y, x) = 0$, but then beginning with $m_4(x, y, y, x)$ and finishing with the higher product $m_{11}(x, y, y, y, y, y, y, y, y, y, x)$, all higher operations of this particular shape have the value z.

This would inspire the following conjecture:

Conjecture 4.3.14. *There is an A_∞-structure on $H^*(D_8, \mathbb{F}_2)$, that can be computed using an implementation of the Kadeishvili algorithm such that for all $n > 3$ we get $m_n(x, y, \dots, y, x) = z$.*

The code used for automating these computations is

```
> for n in [1..7] do
for> for M in Multisets({x,y},n) do
```

```
for|for> S := [];
for|for> for X in M do
for|for|for> Append(~S,X);
for|for|for> end for;
for|for> for ss in [S[I] : I in Permutations({1..#S})] do
for|for|for> tmp:=HighProduct(Aoo,ss);
for|for|for> end for;
for|for> end for;
for> end for;
>  {#k : k in Keys(Aoo`f) | not IsZero(Aoo`f[k]) };
{ 2, 3, 4, 5, 6, 7 }
> {#k : k in Keys(Aoo`f) | not IsZero(Aoo`m[k]) };
{ 2, 3, 4, 5, 6, 7 }
```

The cohomology of D_{16}

The group ring $\mathbb{F}_2 D_{16}$ is given by $\mathbb{F}_2[a,b]/(a^2, b^2, abababab + babababa)$. Its cohomology ring is isomorphic to the cohomology of D_8, and given by $\mathbb{F}_2[x,y,z](xy)$ with $|x| = |y| = 1$ and $|z| = 2$. Hence, just the ring structure on the cohomology ring is not enough to distinguish D_8 from D_{16}. However, the higher multiplication theorem, again, tells us that we can recover the group ring from an A_∞-structure on the cohomology, and thus, we would expect $m_4(x,y,x,y) = 0$ and $m_8(x,y,x,y,x,y,x,y) = z$ on $H^*(D_{16}, \mathbb{F}_2)$. A computation using the MAGMA-tool verifies this.

The cohomology of Q_8

The cohomology ring of the quaternionic unit group has a finite presentation given by $H^*(Q_8, \mathbb{F}_2) = k[x,y,z]/(x^2 + xy + y^2, y^3)$ with $|x| = |y| = 1$ and $|z| = 4$. Just like the cyclic groups, Q_8 has complexity 1, and a periodic resolution on the form, with $\Lambda = \mathbb{F}_2 Q_8$

$$\cdots \to \Lambda^2 \to \Lambda \to \Lambda \to \Lambda^2 \to \Lambda^2 \to \Lambda \to \mathbb{F}_2 \to 0 \quad .$$

The resolution has period 4, and one representative of z is given by the map

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^2 \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda^2 \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda^2 \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Lambda \to \cdots \\
\cdots \arrow[r] & \Lambda^2 \arrow[r] & \Lambda \arrow[r] & \Lambda \arrow[r] & \Lambda^2 \arrow[r] & \Lambda^2 \arrow[r] & \Lambda \arrow[r] & \mathbb{F}_2 \to 0
\end{tikzcd}$$

as can be seen in Example 4.3.15.

Example 4.3.15. The degree 4 generator representative cochain in $H^*(Q_8, \mathbb{F}_2)$

```
> G := ExtraSpecialGroup(2,1:Type:="-");
> Aoo := AInfinityRecord(G,15);
> S<x,y,z> := Aoo`S;
> HighMap(Aoo,[z]);
Basic algebra chain map of degree -4
> ModuleMaps(HighMap(Aoo,[z]));
[*
    // several pages of repeating output removed



    [1 0 0 0 0 0 0 0]
    [0 1 0 0 0 0 0 0]
    [0 0 1 0 0 0 0 0]
    [0 0 0 1 0 0 0 0]
    [0 0 0 0 1 0 0 0]
    [0 0 0 0 0 1 0 0]
    [0 0 0 0 0 0 1 0]
    [0 0 0 0 0 0 0 1],



    [1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1],
```

```
    [1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0]
    [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1],

    [1 0 0 0 0 0 0 0]
    [0 1 0 0 0 0 0 0]
    [0 0 1 0 0 0 0 0]
    [0 0 0 1 0 0 0 0]
    [0 0 0 0 1 0 0 0]
    [0 0 0 0 0 1 0 0]
    [0 0 0 0 0 0 1 0]
    [0 0 0 0 0 0 0 1]
*]
```

Using the function defined as follows, we can detect periodic maps.

```
period := function(f)
  d := Degree(f);
  d1,d2 := Degrees(Domain(f));
  ps := [p : p in [d2-d..d1] | &and [ ModuleMap(f,i) cmpeq
                ModuleMap(f,i+p) : i in [d2-d..d1-p] ] ];
return Minimum(ps);
end function;
```

Now, since the coclass z has a representative consisting of identity maps, and since the minimal resolution we're working with is periodic, we might be able to use the results in Lemma 4.3.3. It all boils down to whether the higher components of the quasi-isomorphism stay periodic with a good period. Specifically, periods 1, 2 and 4 are good, and anything else derails our first approach to the computational process.

Some computations quickly reveal a problem:

```
> G := ExtraSpecialGroup(2,1:Type:="-");
> Aoo := AInfinityRecord(G,64);
> S<x,y,z> := Aoo`S;
> period(HighMap(Aoo,[z]));
4 1
> period(HighMap(Aoo,[x]));
4 1
> period(HighMap(Aoo,[y]));
4 1
> period(HighMap(Aoo,[x,x]));
8 1
> period(HighMap(Aoo,[x,x,x]));
8 1
> period(HighMap(Aoo,[x,x,x,x]));
16 1
> period(HighMap(Aoo,[x,x,x,x,x]));
16 1
> period(HighMap(Aoo,[x,x,x,x,x,x]));
16 1
> period(HighMap(Aoo,[x,x,x,x,x,x,x]));
16 1
> period(HighMap(Aoo,[x,x,x,x,x,x,x,x]));
32 1
> period(HighMap(Aoo,[x,x,x,x,x,x,x,x,x]));
32 1
> period(HighMap(Aoo,[x,x,x,x,x,x,x,x,x,x]));
32 1
```

From this computation, we can read off that all f_1 are reasonable. However, already $f_2(x,x)$ has a too high period, leading to non-commutativity with $f_1(z)$ and hence demonstrating that the results in Lemma 4.3.3 can not be applied. This provokes the questions:

Question 4.3.16. Can we provoke arbitrarily high periods of the higher homotopies in an A_∞-structure on $H^*(Q_8,\mathbb{F}_2)$?

Question 4.3.17. Is there some choice of representatives and homotopies such that there is an upper bound on the periods of the homotopies in an A_∞-structure on $H^*(Q_8,\mathbb{F}_2)$?

I suspect, however, that the answers to these questions is not what would make the computational life the easiest. Hence the following conjectured answers to the questions:

Conjecture 4.3.18. *All A_∞-structures on $H^*(Q_8,\mathbb{F}_2)$ have $f_2(x,x)f_1(z) \neq f_1(z)f_2(x,x)$.*

Conjecture 4.3.19. *All A_∞-structures on $H^*(Q_8, \mathbb{F}_2)$, with quasi-isomorphism $f_*\colon H^*(Q_8, \mathbb{F}_2) \to \mathrm{End}_{\mathbb{F}_2 Q_8}(p\mathbb{F}_2, p\mathbb{F}_2)$ has elements of arbitrarily high periodicity in the image of f_*.*

It is worth noting that these observations indicate that Lemma 4.3.3 is only really applicable for the cyclic p-groups. Indeed, p-groups of complexity 1 are either cyclic or generalized quaternionic, and these computations indicate that Lemma 4.3.3 is probably not applicable for the generalized quaternionic groups.

For the case of an abelian group of order divisible by at least two primes, only the Sylow subgroups are visible in cohomology, as any direct summand of order not divisible by p will correspond to a tensor factor k in cohomology.

Appendix A

Implementation details

> Code that only you can understand does not make you an advanced coder; code that your grandmother can understand does!
>
> Jeremy Brown
> *Macromedia Flash MX 2004 ActionScript 2.0 Dictionary*

A.1 Diagonals and Haskell

Here, we include the source code of the Haskell module computing the Saneblidze-Umble diagonal.

```haskell
module SaneblidzeUmbleSigns where
import Data.List
import Data.Maybe
import qualified Data.Map as Map
import Data.Map ((!))

type Sequence  = [Int]
type Partition = [Sequence]

monotonicSequence :: (a -> a -> Bool) -> [a] -> [[a]]
monotonicSequence _ []          = []
monotonicSequence _ [x]         = [[x]]
monotonicSequence cmp (x:y:etc) =
    if   x `cmp` y
    then (x:s):ss
```

```
    else  [x]:(s:ss)
  where
    (s:ss) = monotonicSequence cmp (y:etc)

rising :: Sequence -> Partition
rising  = monotonicSequence (<=)

falling :: Sequence -> Partition
falling  = monotonicSequence (>=)

type Face = (Partition, Partition)
type SignFace = (Int, Partition, Partition)

stripSign :: SignFace -> Face
stripSign (s,p,q) = (p,q)

buildFace :: Sequence -> SignFace
buildFace p = signFace (map sort (falling p), reverse (rising p))

orbit :: Int -> Sequence -> Sequence
orbit a pi = findOrbit a []
  where
    findOrbit a as =
      if    a' `elem` as
      then  (sort.nub) (a:as)
      else  findOrbit a' (a:as)
          where
            a' = pi !! (a-1)

pSign :: Sequence -> Int
pSign pi = signPi
  where
    getOrbits orbs []      = orbs
    getOrbits orbs (p:ps)  = getOrbits (o:orbs) (ps \\ o)
      where
        o = orbit p pi
    orbits       = getOrbits [] pi
    orbitLengths = map length orbits
    evenCycles   = filter even orbitLengths
    signPi       = (-1)^(length evenCycles)

signR :: Partition -> Int
signR q = (-1)^epsilon * (pSign pi)
```

```
  where
    pi          = concat q
    epsilon     = sum summands
    summands    = map (\i -> i * (qLengths !! (i-1)))
                      [1..((length q) - 1)]
    qLengths    = map length q

orSign :: Partition -> Int
orSign p = (-1)^exponent
  where
    exponent        = exponent2 `div` 2
    exponent2       = (sum lengthSquares) -
                      ((length . concat) p)
    lengthSquares   = map ((^2) . length) p

signFace :: Face -> SignFace
signFace (p,q) = (qSign * rSign * sign1, p, q)
  where
    qSign       = (-1)^qExp
    qExp        = (choose2 . length) q
    rSign       = orSign  p
    sign1       = signR   (reverse q)
    choose2 n   = n*(n-1) `div` 2

showSignFace :: SignFace -> String
showSignFace f@(s,_,_) =
  case s of
  1   -> "+" ++ (showFace . stripSign) f
  -1  -> "-" ++ (showFace . stripSign) f
  0   -> ""
  a   -> (show a) ++ "." ++ (showFace . stripSign) f

showFace  :: Face -> String
showFace = showFaceTemplate showPartition

showFaceShort :: Face -> String
showFaceShort = showFaceTemplate showPartitionShort

showFaceTemplate :: (Partition -> String) -> Face -> String
showFaceTemplate showP (u,v) = showP u ++ "x" ++ (showP . reverse) v

showPartitionShort :: Partition -> String
showPartitionShort = filter (/=',') . showPartition
```

```
showPartition :: Partition -> String
showPartition p = pString p
  where
    pString = concat . intersperse "|" . partsStrings
    partsStrings  = map (concat . intersperse "," . map show)

showMatrix :: Face -> String
showMatrix (f1,f2) = unlines $ map (showLine f1) f2
  where
    showLine   a b = concatMap (flip showPoint b) a
    showPoint  a b =
      if    intersect a b /= []
      then  show $ head $ intersect a b
      else  "."

permutations :: Int -> [Sequence]
permutations n = permuteList [1..n]
  where
    permuteList [] = []
    permuteList [a] = [[a]]
    permuteList ls = concatMap
                       (\x -> map (x:) (permuteList (ls \\ [x] )))
                       ls

isAdmissible :: SignFace -> [Int] -> Bool
isAdmissible f@(_,pi,mu) m = admitted
  where
    mIntersectPi    = map (intersect m) pi
    partsWithM      = findIndices ((==length m).length)
                                     mIntersectPi
    mInUniquePart   = (1 == length partsWithM)
    j               = head partsWithM
    jLessK          = j < length pi
    pi_j            = pi !! j
    pi_j1           = pi !! (j+1)
    lmLesslpij      = length m < length pi_j
    properSubset    = mInUniquePart && jLessK && lmLesslpij
    minLargerMax    = (minimum m > maximum pi_j1)
    k               = fromJust (findIndex (minimum m `elem`) mu)
    mus             = concat (drop k mu)
    allzero         = null (intersect pi_j1 mus)
    admitted        = properSubset && minLargerMax && allzero
```

```
moveSubset :: SignFace -> [Int] -> Maybe SignFace
moveSubset f@(_,p',_) m =
  if     isAdmissible f m
  then   Just (foldl' moveElement f (sort m))
  else   Nothing
    where
      moveElement :: SignFace -> Int -> SignFace
      moveElement (s,p,q) e = (s',p',q)
        where
          (Just i)  = findIndex (e `elem`) p
          pi        = p !! i
          pi1       = p !! (i+1)
          pmoved    = (take i p) ++
                        [pi\\[e],pi1++[e]] ++
                        (drop (i+2) p)
          lowercut  = filter (>e) pi
          uppercut  = filter (<e) pi1
          expmoved  = length (lowercut ++ uppercut)
          (s',p')   =
            if    e > maximum pi1
            then  (-s*(-1)^expmoved, pmoved)
            else  (s,p)

admissiblesInPin :: Int -> SignFace -> [[Int]]
admissiblesInPin i f@(_,pi,_)  = filter (not.null) admissibleSets
  where
    admissibleSets =
      if i+2 > length pi
      then []
      else filter  (isAdmissible f) candidates
        where
          candidates      = filter (not . null) (subsets large)
          large           = filter (>m) (pi !! i)
          m               = maximum (pi !! (i+1))

subsets :: [a] -> [[a]]
subsets []      = [[]]
subsets (a:as)  = map (a:) (subsets as) ++ subsets as

twist :: SignFace -> SignFace
twist (s,a,b) = (s,b,a)
```

```
derivedFaces :: SignFace -> [SignFace]
derivedFaces f@(_,p,q) = derivedRightQ 0 [] [f]
  where
    lp = length p
    lq = length q
    derivedRightQ i r [] =
      if    i >= lp-2
      then  derivedDownQ 0 [] r
      else  derivedRightQ (i+1) [] r
    derivedRightQ i r (s:ss) = derivedRightQ i (s:r) (ss++rights)
      where
        rights = mapMaybe (moveSubset s) (admissiblesInPin i s)
    derivedDownQ i d [] =
      if    i >= lq-2
      then  d
      else  derivedDownQ (i+1) [] d
    derivedDownQ i d (s:ss) = derivedDownQ i (s:d) (ss++downs)
      where
        s'            = twist s
        downs         = map twist twistedDowns
        twistedDowns  = mapMaybe (moveSubset s')
                            (admissiblesInPin i s')

type LinearCombination vectors = Map.Map vectors Int

showLinearCombination :: LinearCombination Face -> String
showLinearCombination lc = concatMap showSignFace (signFaceList lc)

addSignFaces :: [SignFace] -> LinearCombination Face
addSignFaces [] = Map.empty
addSignFaces ((s,p,q):as) = Map.insertWith (+) (p,q) s
                                (addSignFaces as)

signFaceList :: LinearCombination Face -> [SignFace]
signFaceList lc = map (\ ((p,q),s) -> (s,p,q)) (Map.toList lc)

permutahedronDiagonal :: Int -> LinearCombination Face
permutahedronDiagonal n = (addSignFaces . nub) deriveds
  where
    deriveds = concatMap derivedFaces primitiveFaces
    primitiveFaces = map buildFace (permutations n)

derivedConsecutive :: Partition -> Bool
```

```
derivedConsecutive pi = checkPartition [] pi
  where
    checkPartition n []          = True
    checkPartition n (pij:pi')  =
      if    (intersect n' range == range)
      then  checkPartition n' pi'
      else  False
      where
        n'     = sort (n ++ pij)
        range  = [minimum pij .. maximum pij]

associahedronDiagonal :: Int -> LinearCombination Face
associahedronDiagonal n =
  Map.filterWithKey checkFace (permutahedronDiagonal n)
  where
    checkFace (f1,f2) s =
      derivedConsecutive f1 && derivedConsecutive (reverse f2)
```

A.2 Black-box computation and Magma

The code for computing A_∞-structures in group cohomology is released with Magma version 2.14. The code builds on top of the cohomology ring computation code by Jon F. Carlson, and follows the Kadeishvili algorithm closely.

In order to use it, we need to set up a data structure carrying cached computed higher multiplications and fragments of a quasi-isomorphism as well as the group algebra, the cohomology ring and a projective resolution. The length of the initial part of the resolution needed to compute with must be given explicitly – and unless it is enough to compute the entire cohomology ring, no guarantees can be made for correctness of the computations. However, recognition of sufficient length is built into Magma.

A typical session looks like the following:

```
> G := DihedralGroup(8);
> Aoo := AInfinityRecord(G,15);
> S<x,y,z> := Aoo`S;
> HighProduct(Aoo,[x,y,x,y]);
z
```

There is also the command HighMap with the same kind of arguments as HighProduct, which returns the corresponding images of the quasi-isomorphism $\mathrm{Ext}^*_R(k,k) \to \mathrm{End}^*_R(pk)$ as Magma chain maps from the projective resolution in AooʻP to itself.

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